

Time Reversal Invariance in the β -Decays of $A = 8$ Nuclei

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Abstract

We examine time reversal invariance in the β -decays of ${}^8\text{B}(2^+)$ and ${}^8\text{Li}(2^+)$ to ${}^8\text{Be}(2^+)$ in detail, with particular attention to final state interactions of the two α particles from the decay of ${}^8\text{Be}$, and of the electron (positron) with the daughter nucleus. An R-matrix formulation is used, with the initial state described by a shell model. The R-matrix parameters are obtained by fitting the $\alpha - \alpha$ scattering phase shifts and allowed β -decay rates. Because the nuclear final state interaction effects on the time reversal test come from second forbidden β -decays, they are small and can be minimized by a suitable choice of kinematic conditions. The e^\pm -nucleus Coulomb interaction induced T-violation effects depend on the first forbidden operators.

1 Introduction

Tests of time reversal invariance (TRI) remain very important because it is only in the $K^0 - \bar{K}^0$ system that CP (or T) invariance has been found not to

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hold to about 2×10^{-3} [1]. Until TRI violations (TRIV) have been found in other systems, it is difficult to distinguish between various theoretical proposals [2-7] that have been put forward to explain the experimental results obtained in the decays of neutral kaons.

Nuclear tests of TRI that have been carried out include detailed balance [8] ($a_T \leq 5 \times 10^{-4}$), reciprocity [9] ($a_T \leq 10^{-2}$), polarized hyperon decay [10, 11], nuclear γ -decay [12] and nuclear β -decay [13] ($a_T \leq 10^{-3}$) experiments. The most accurate test of TRI, combined with parity non-conservation is the lack of neutron and atomic electric dipole moments [1, 14]. For a review of experiments and theory, we refer to Ref. [11, 15].

Tests of TRI in β -decay are made difficult by the necessity of measuring a triple correlation or the polarization of the emitted electron. The triple correlation $\langle \vec{J} \rangle \cdot \vec{p}_e \times \vec{p}_\nu$ requires a polarized parent state and the measurement of the recoil nucleus. The mass eight system is amenable to a somewhat simpler test, because the daughter of the decays from ${}^8\text{Li}$ or ${}^8\text{B}$, namely ${}^8\text{Be}^*$, is unstable and breaks up into two charged particles (${}^4\text{He}$) which can be detected more easily than the recoil. However, a question arises immediately: Does the final state interaction amongst the final e^\pm and the two α particles emitted in the decay of ${}^8\text{Be}^*$ spoil the TRI test? In this work we investigate this question in detail. We find that true tests of TRI which depend on measurements of one (or both) recoil α particles are only spoiled by competing second order forbidden β -decays and by the e^\pm -nucleus Coulomb interactions. By a proper choice of kinematics, even this small spoiling can be minimized. The primary decay is from a 2^+ state of ${}^8\text{Li}$ or ${}^8\text{B}$ to the lowest 2^+ state of ${}^8\text{Be}^*$. “Pseudo-TRI violation” (PTRIV) for the strong interaction effects, which mimics TRIV through the introduction of phases [16] occur primarily through the interference of the main $2^+ \rightarrow 2^+$ matrix elements with those of the $2^+ \rightarrow 0^+$ and $2^+ \rightarrow 4^+$ ($J = 0^+$ and 4^+) of ${}^8\text{Be}$; the PTRIV effects for the e^\pm -nucleus Coulomb interaction depend on the Coulomb phase shifts and the $2^+ \rightarrow 2^+$ transition operators.

A complete first-principle theoretical study is immensely complex. We treat the decay as a two-step process (see Fig. 1)

$${}^8\text{Li}({}^8\text{B}) \rightarrow {}^8\text{Be}^* + e^-(e^+) + \bar{\nu}(\nu)$$

$$\rightarrow \alpha + \alpha + e^-(e^+) + \bar{\nu}(\nu),$$

and use an R-matrix formalism in which the initial state is described by a shell model wave function and the ${}^8\text{Be}^*$ resonant states are saturated by shell model states. The parameters of the R-matrix are constrained by physical considerations and are adjusted to reproduce α - α scattering phase shifts in the relevant energy region. With these parameters, it is found that the unpolarized β -decay spectrum is reproduced with two additional parameters. The TRIV and PTRIV correlation observables are studied in this framework.

In the next section, we examine the TRI tests in the mass 8 system. In Section 3, we develop the R-matrix approximation to be used. In Section 4, we study the PTRIV due to hadronic interactions; in Section 5 we study that due to the e^\pm -nucleus Coulomb interactions, and finally in Section 6 we summarize our results.

2 Time reversal invariance tests in the β -decay processes of the A=8 System

The questions related to tests of TRI in the weak and electromagnetic processes have been studied by many authors. We restrict ourselves here to the mass 8 system, for which an experimental study has been proposed [17]. The initial 2^+ ${}^8\text{Li}$ or ${}^8\text{B}$ ground states decay by β -emission to the (resonant) states of ${}^8\text{Be}^*$ with $J^P = 0^+, 2^+, 4^+, \dots$, which break up into two α particles. The graphical representation, together with definitions of some kinematic variables are given in Fig. 1. The final 2^+ resonant states of ${}^8\text{Be}^*$ are our main interest. Those states with $J^P = 0^+$ and 4^+ can interfere with the 2^+ state to produce PTRIV signals due to final (2α) state interactions. These states and interactions should be taken into account in the study of TRIV effects. The resonant states of ${}^8\text{Be}^*$ are treated in section 3. In the present section, emphasis is given to presenting the analytic results and the shell model computation of the hadronic weak transition amplitudes.

a) The decay rates for the unpolarized case

The differential decay rate for the β -decay of an unpolarized nucleus can be expressed as

$$dW = 2\pi\delta(E_i - E_f - \epsilon_e - \epsilon_\nu) Tr [H_W^\dagger \Theta H_W] d\xi, \quad (2.1)$$

where $d\xi$ is an infinitesimal phase space element of the final state, Θ is the final state projection operator, H_W is the weak Hamiltonian responsible for the weak charged current nuclear reactions and the trace is over the initial and final state (spin + linear momentum) subspaces. We write

$$Tr [H_W^\dagger \Theta H_W] = \frac{1}{2} G_F^2 \cos^2 \theta_c W^{\mu\nu} L_{\mu\nu}. \quad (2.2)$$

When the leptonic polarizations are not detected, $W^{\mu\nu}$ and $L^{\mu\nu}$ are

$$W^{\mu\nu} = Tr [J^{\nu\dagger}(-\mathbf{q}) \Theta J^\mu(-\mathbf{q})], \quad (2.3)$$

$$L^{\mu\nu} = Tr [l^{\nu\dagger}(\mathbf{q}) l^\mu(\mathbf{q})], \quad (2.4)$$

where $J^\mu(\mathbf{q})$ is the Fourier transform of the hadronic weak current operator $J^\mu(\mathbf{x}, t = 0)$, $l_\mu(\mathbf{q})$ is Fourier transform of the matrix element of the leptonic weak current operators with $\mathbf{q} = -\mathbf{k}_e - \mathbf{k}_\nu$.^{2.1} Eq. 2.1 can be written as

$$dW = 2\pi\delta(E_i - E_f - \epsilon_e - \epsilon_\nu) \frac{1}{2} G_F^2 \cos^2 \theta_c [W_{ss} L_{ss} + W_{sv}^{(+i)} L_{sv}^{(+i)} + W_{sv}^{(-i)} L_{sv}^{(-i)} + W_{vv}^{ij} L_{vv}^{ij}] d\xi, \quad (2.5)$$

where $i, j = 1 \dots 3$ and irreducible tensor components of the leptonic tensors L_{ss}, \dots, L_{vv} are given in the following subsection. W_{ss} and W_{vv} contain contributions of the time (labeled by “s”) and space (labeled by “v”) components of the hadronic weak current operators. $W_{sv}^{(\pm)}$ contains contributions of the interference terms between the time and space components of the same current operator, where “+” (“−”) denotes the symmetric (anti-symmetric) combination with respect to the time and space indices of the corresponding operator.

^{2.1}The Coulomb interactions between the charged leptons and nuclei that are ignored here will be investigated in Section 5.

b) Kinematic functions and hadronic response functions

In the decay of ${}^8\text{B}$ and ${}^8\text{Li}$, the daughter ${}^8\text{Be}^*$ breaks up into two α particles. The resonant state of the two α particles is of the following form

$$| -; \hat{\mathbf{k}} \rangle_{2\alpha} = 4\pi \sum_{J_f=0^+, 2^+, 4^+} \sum_{M_f=-J_f}^{J_f} i^{J_f} e^{-i\delta_{J_f}} Y_{J_f M_f}^*(\hat{\mathbf{k}}) | J_f M_f \rangle, \quad (2.6)$$

where “ $-$ ” denotes an incoming boundary condition, $\hat{\mathbf{k}}$ is a unit vector in the direction of the relative momentum of the two α particles, δ_{J_f} is the strong and electromagnetic phase shift of the J_f partial wave and Y_{JM} is a spherical harmonic.

Using the trace formula given in [18], various terms in Eq. 2.5 can be expressed in terms of a set of irreducible hadronic response functions, namely,

$$W_{ss} L_{ss} = (4\pi)^2 \sum_{J_f J'_f} i^{J'_f - J_f} e^{i(\delta_{J_f} - \delta_{J'_f})} \sum_{\sigma} C_{\sigma}^{J_f J'_f} R_{ss}(\sigma) K_{ss}(\sigma), \quad (2.7)$$

$$W_{sv}^{(\pm)i} L_{sv}^{(\pm)i} = (4\pi)^2 \sum_{J_f J'_f} i^{J'_f - J_f} e^{i(\delta_{J_f} - \delta_{J'_f})} \sum_{\sigma \rho} C_{\sigma}^{J_f J'_f} R_{sv}^{(\pm)}(\sigma \rho) K_{sv}^{(\pm)}(\sigma \rho), \quad (2.8)$$

$$W_{vv}^{ij} L_{vv}^{ij} = (4\pi)^2 \sum_{J_f J'_f} i^{J'_f - J_f} e^{i(\delta_{J_f} - \delta_{J'_f})} \sum_{\sigma \rho \tau} C_{\sigma}^{J_f J'_f} R_{vv}(\sigma \rho \tau) K_{vv}(\sigma \rho \tau), \quad (2.9)$$

where

$$C_{\sigma}^{J_f J'_f} = \sqrt{4\pi} \frac{\langle J'_f || Y_{\sigma} || J_f \rangle}{\tilde{J}_i \tilde{\sigma}} \quad (2.10)$$

with $\tilde{x} \equiv \sqrt{2x+1}$. The kinematic functions are

$$K_{ss}(\sigma) = 4\pi [Y_{\sigma}(\hat{\mathbf{k}}) \otimes Y_{\sigma}(\hat{\mathbf{q}})]_0 L_{ss}, \quad (2.11)$$

$$K_{sv}^{(\pm)}(\sigma \rho) = 4\pi [Y_{\sigma}(\hat{\mathbf{k}}) \otimes [Y_{\rho}(\hat{\mathbf{q}}) \otimes L_{sv}^{(\pm)}]_{\sigma}]_0, \quad (2.12)$$

$$K_{vv}(\sigma \rho \tau) = 4\pi [Y_{\sigma}(\hat{\mathbf{k}}) \otimes [Y_{\rho}(\hat{\mathbf{q}}) \otimes L_{vv}^{\tau}]_{\sigma}]_0, \quad (2.13)$$

where $[\phi_{l_1} \otimes \phi_{l_2}]_{jm}$ denotes the Clebsch–Gordan coupling of $\phi_{l_1 m_1}$ and $\phi_{l_2 m_2}$ to total angular momentum j and magnetic quantum number m . The total number of different kinematic functions are 34; 22 of them are T–even and 12 of them are T–odd.

It can be shown that the leading terms of the hadronic response functions $R_{sv}^{(\pm)}(\sigma\rho)$ and $R_{vv}(\sigma\rho\tau)$ are of order $O[(R\kappa)^\rho]$ (it is $O[(R\kappa)^\sigma]$ for $R_{ss}(\sigma)$), where R is a typical radius of the parent/daughter nucleus; thus at low momentum transfer, terms with large ρ can be neglected since $R\kappa \ll 1$. If one keeps terms up to order $O[\kappa^2 R^2]$, the possible kinematic functions with $\sigma \leq 2$ together with their properties under parity and time reversal transformations are listed in Table 1. The T–odd correlation observables interested in this paper are P–odd. There are four of them which are given in the lower right box of Table 1.

The leptonic tensors in case of not observing the polarization of the charged leptons are found to be

$$L_{ss} = 8 [\epsilon_e \epsilon_\nu + \mathbf{k}_e \cdot \mathbf{k}_\nu], \quad (2.14)$$

$$L_{sv}^{(+)} = 8 [\epsilon_e \mathbf{k}_\nu + \epsilon_\nu \mathbf{k}_e], \quad (2.15)$$

$$L_{sv}^{(-)} = \mp 8i \mathbf{k}_e \times \mathbf{k}_\nu, \quad (2.16)$$

$$L_{vv}^0 = -\frac{8}{\sqrt{3}} [3\epsilon_e \epsilon_\nu - \mathbf{k}_e \cdot \mathbf{k}_\nu], \quad (2.17)$$

$$L_{vv}^1 = \pm 8\sqrt{2} [\epsilon_\nu \mathbf{k}_e - \epsilon_e \mathbf{k}_\nu], \quad (2.18)$$

$$L_{vv}^2 = 8 \left[\mathbf{k}_e \mathbf{k}_\nu + \mathbf{k}_\nu \mathbf{k}_e - \frac{2}{3} \mathbf{k}_e \cdot \mathbf{k}_\nu \right]. \quad (2.19)$$

The hadronic dynamic response functions R_{ss}, \dots, R_{vv} can be expressed in terms of the reduced matrix elements of the multipole operators [19] C_J, \dots, M_J^5 of the hadronic charged weak currents. The analytic form of these relations can be found in Appendix A. The T–odd response functions, which are of interest in this work, have the following explicit expressions.

Contribution from the transition to the $J^P = 2^+$ states:

$$R_{sv}^{(+)}(22) = i\frac{2}{5} \text{Im} C_0 M_2^{5*} + i\frac{1}{5} \sqrt{\frac{3}{2}} \text{Re} C_2 E_1^{5*}, \quad (2.20)$$

$$\begin{aligned}
R_{sv}^{(-)}(21) = & -i\frac{2}{5}\sqrt{\frac{3}{5}}ImC_0E_2^* - i\frac{2}{5}\sqrt{\frac{2}{5}}ImC_0L_2^* - i\frac{2}{25}\sqrt{7}ImL_1^5C_1^{5*} \\
& -i\frac{3}{25}\sqrt{\frac{7}{2}}ImE_1^5C_1^{5*} + i\frac{2}{5}\sqrt{\frac{2}{5}}ImL_0C_2^*, \tag{2.21}
\end{aligned}$$

$$\begin{aligned}
R_{vv}(221) = & i\frac{1}{10}\sqrt{\frac{21}{5}}ImL_1^5E_1^{5*} - i\frac{\sqrt{2}}{5}ImL_0E_2^* + i\frac{2}{5}\sqrt{\frac{3}{35}}ImL_3^5E_1^{5*} \\
& -i\frac{4}{5}\sqrt{\frac{2}{35}}ImL_1^5E_3^{5*} + i\frac{1}{10}ReL_1^5M_2^{5*}, \tag{2.22}
\end{aligned}$$

$$\begin{aligned}
R_{vv}(212) = & -i\frac{\sqrt{21}}{50}ImL_1^5M_1^* - i\frac{2}{25}\sqrt{\frac{21}{2}}ImE_1^5M_1^* + i\frac{1}{5}\sqrt{\frac{2}{5}}ImL_0M_2^{5*} \\
& + i\frac{1}{5}\sqrt{\frac{2}{5}}ReE_1^5E_2^* + i\frac{1}{10}\sqrt{\frac{3}{5}}ReE_1^5L_2^* - i\frac{1}{10\sqrt{5}}ReL_1^5E_2^*. \tag{2.23}
\end{aligned}$$

Here, symbols with a superscript “5” indicate that they are contributions of the hadronic axial vector current operator; others are contributions of the hadronic vector current operator. C_J or C_J^5 are reduced matrix elements of the multipole operators of the weak charge density operators. $L_J \dots M_J^5$ are matrix elements of the space components of the weak current operators. L_J or L_J^5 are reduced matrix elements of the multipole operators of the longitudinal part of the weak current operators. E_J or E_J^5 are reduced matrix elements of the electric part of the transverse components of the weak current multipole operators and M_J or M_J^5 are the reduced matrix elements of the magnetic part of the transverse components of the same operators.

Contribution from the interference terms between $J^P = 2^+$ and $J^P = 0^+$ states:

$$R_{sv}^{(+)}(22) = \frac{1}{5}C_0M_2^{5*} - \frac{1}{5}\sqrt{\frac{3}{2}}E_1^5C_2^* + \frac{\sqrt{2}}{10}C_1E_2^*, \tag{2.24}$$

$$R_{sv}^{(-)}(21) = -\frac{1}{5}\sqrt{\frac{3}{5}}C_0E_2^* + \frac{1}{5}\sqrt{\frac{2}{5}}L_0C_2^* - \frac{1}{5}\sqrt{\frac{2}{5}}C_0L_2^*, \tag{2.25}$$

$$R_{vv}(221) = -\frac{\sqrt{2}}{10}L_0E_2^* - \frac{1}{10}L_1^5M_2^{5*}, \tag{2.26}$$

$$R_{vv}(212) = \frac{1}{10}\sqrt{\frac{2}{5}}L_0M_2^{5*} - \frac{1}{5}\sqrt{\frac{2}{5}}E_1^5E_2^* - \frac{1}{10}\sqrt{\frac{3}{5}}E_1^5L_2^*$$

$$+\frac{\sqrt{5}}{50}L_1^5E_2^*, \quad (2.27)$$

where those reduced matrix elements that are complex conjugated correspond to transitions to the final 0^+ state and all the rest correspond to the transition to the final 2^+ state.

Contribution from the interference terms between $J^P = 2^+$ and $J^P = 4^+$ states:

$$R_{sv}^{(+)}(22) = \frac{1}{5}C_0M_2^{5*} + \frac{1}{5}\sqrt{\frac{2}{5}}E_1^5C_2^* - \frac{\sqrt{2}}{15}C_1E_2^*, \quad (2.28)$$

$$R_{sv}^{(-)}(21) = -\frac{1}{5}\sqrt{\frac{3}{5}}C_0E_2^* + \frac{1}{5}\sqrt{\frac{2}{5}}L_0C_2^* - \frac{1}{5}\sqrt{\frac{2}{5}}C_0L_2^*, \quad (2.29)$$

$$R_{vv}(221) = -\frac{\sqrt{2}}{10}L_0E_2^* + \frac{1}{15}L_1^5M_2^{5*} \quad (2.30)$$

$$R_{vv}(212) = \frac{1}{10}\sqrt{\frac{2}{5}}L_0M_2^{5*} + \frac{2}{15}\sqrt{\frac{2}{5}}E_1^5E_2^* + \frac{1}{25}\sqrt{\frac{5}{3}}E_1^5L_2^* - \frac{\sqrt{5}}{75}L_1^5E_2^*, \quad (2.31)$$

where those reduced matrix elements that are complex conjugated correspond to transitions to the final 4^+ state and all the rest correspond to the transition to the final 2^+ state.

The T-odd observables corresponding to these interference response functions depend on their imaginary parts.

The T-even response functions can also be written in the same expanded form as those given above for the T-odd ones. They will not be presented here; their expressions can be deduced by using the compact expressions given in Appendix A.

c) The differential decay rate in terms of the weak transition amplitudes

The kinematic functions can be written in Cartesian form. The differential decay rate Eq. 2.5 in the 2^+ resonant region is

$$dW = dW_{22}^{(e)} + dW_{22}^{(o)} + dW_{20}^{(o)} + dW_{24}^{(o)} + \dots, \quad (2.32)$$

where $dW_{22}^{(e)}$ is a contribution to the differential decay rate even under time reversal and $dW_{JJ'}^{(o)}$ are contributions to differential decay rates odd under time reversal. The subindices denote the contributing angular momentum (J) of the partial waves of the final two α particles. The relatively small terms that originate from the interference between the 2^+ and the 0^+ and 4^+ final states are included for the T-odd observables only since they are important contributions to the PTRIV that originate from the final state interaction. The contributions from pure 0^+ and 4^+ states to the TRIV signal are negligible in the energy region where the lowest 2^+ state resonates; they are not included in the differential decay rate.

When m_e is neglected, the T-even differential decay rate has the following generic form

$$\begin{aligned} dW_{22}^{(e)} = dW_0 & \left\{ \left[R_1^{(0)} + R_2^{(0)} \hat{\mathbf{k}}_e \cdot \hat{\mathbf{k}}_\nu + R_3^{(0)} \left((\hat{\mathbf{k}}_e \cdot \hat{\mathbf{k}}_\nu)^2 - \frac{1}{3} \right) \right] \right. \\ & + T_2(\hat{\mathbf{k}}) : \left[\left(R_1^{(2)} \hat{\mathbf{k}}_e \hat{\mathbf{k}}_e + R_2^{(2)} \hat{\mathbf{k}}_e \hat{\mathbf{k}}_\nu + R_3^{(2)} \hat{\mathbf{k}}_\nu \hat{\mathbf{k}}_\nu \right) \right. \\ & \left. \left. + \left(R_4^{(2)} \hat{\mathbf{k}}_e \hat{\mathbf{k}}_e + R_5^{(2)} \hat{\mathbf{k}}_e \hat{\mathbf{k}}_\nu + R_6^{(2)} \hat{\mathbf{k}}_\nu \hat{\mathbf{k}}_\nu \right) \hat{\mathbf{k}}_e \cdot \hat{\mathbf{k}}_\nu \right] \right\}, \quad (2.33) \end{aligned}$$

where the leading decay rate dW_0 is

$$dW_0 = (2\pi)^3 \delta(E_i - E_f - \epsilon_e - \epsilon_\nu) \frac{16}{5} \epsilon_e \epsilon_\nu G_F^2 \cos^2 \theta_c F(Z, \epsilon_e) |g_A|^2 |A_2|^2 d\xi, \quad (2.34)$$

and the non-relativistic one body Gamow-Teller matrix element A_2 is defined as

$$A_2 = \langle 2^+ || \mathcal{Y}_{20} \cdot \sigma || 2^+ \rangle \quad (2.35)$$

with $\mathcal{Y}_{Jlm} = [Y_l \otimes \hat{\epsilon}]_{Jm}$, $\hat{\epsilon}^\dagger \cdot \hat{\epsilon} = 1$ and $F(Z, \epsilon_e)$ the Coulomb function for the charged leptons. In Eq. 2.34, the tensorial contraction between the second rank tensor $T_2(\hat{\mathbf{k}})$ and a pair of vectors \mathbf{AB} is defined as

$$T_2(\hat{\mathbf{k}}) : \mathbf{AB} \equiv T_2(\hat{\mathbf{k}})_{ij} A_i B_j, \quad (2.36)$$

$$T_2(\hat{\mathbf{k}})_{ij} = \hat{k}_i \hat{k}_j - \frac{1}{3} \delta_{ij}. \quad (2.37)$$

The neglect of m_e is justifiable since the energy released in the transitions ($\sim 3\text{--}10$ MeV) is much larger than m_e .

Three T-odd differential rates are

$$dW_{22}^{(0)} = dW_0 T_2(\hat{\mathbf{k}}) : \left[R_7^{(2)} \hat{\mathbf{k}}_e \hat{\mathbf{k}}_e \times \hat{\mathbf{k}}_\nu + R_8^{(2)} \hat{\mathbf{k}}_\nu \hat{\mathbf{k}}_e \times \hat{\mathbf{k}}_\nu \right], \quad (2.38)$$

$$dW_{20}^{(0)} = dW_0 T_2(\hat{\mathbf{k}}) : \left[R_9^{(2)} \hat{\mathbf{k}}_e \hat{\mathbf{k}}_e \times \hat{\mathbf{k}}_\nu + R_{10}^{(2)} \hat{\mathbf{k}}_\nu \hat{\mathbf{k}}_e \times \hat{\mathbf{k}}_\nu \right], \quad (2.39)$$

$$dW_{24}^{(0)} = dW_0 T_2(\hat{\mathbf{k}}) : \left[R_{11}^{(2)} \hat{\mathbf{k}}_e \hat{\mathbf{k}}_e \times \hat{\mathbf{k}}_\nu + R_{12}^{(2)} \hat{\mathbf{k}}_\nu \hat{\mathbf{k}}_e \times \hat{\mathbf{k}}_\nu \right]. \quad (2.40)$$

where we also have neglected m_e .

The Cartesian response functions $R_i^{(0)}$ ($i = 1, \dots, 3$) and $R_i^{(2)}$ ($i = 1, \dots, 12$) are related to those in the spherical harmonics basis [18], namely R_{ss}, \dots, R_{vv} . They are given in Appendix B. At low momentum transfers, we expand these momentum dependent multipole operators of the hadronic charged weak currents in terms of their static multipoles, up to order of $O(\kappa/M)$ with M the mass of a nucleon, and/or $O(R^2\kappa^2)$ with $R \sim 1 - 2$ fm the size of a typical nucleus. As a first step, we take the non-relativistic one body approximation for the hadronic charged weak current operators by retaining only the leading $\frac{1}{M}$ term in an expansion. The justification of such a step can be understood following the discussions of Ref. [20]. The results are well known [19]. They are reproduced in Appendix C for completeness.

The result of the expansion of the hadronic charged weak currents operator in the non-relativistic one body approximation and the definition of the related static multipole operators are given in Tables 2 and 3. The symbol $\langle || \dots || \rangle$ represents the reduced matrix elements.

It is useful to introduce the following quantities

$$\eta_i = \frac{A_i}{A_2}, \quad (2.41)$$

with A_i defined in Table 3, and

$$f_1 = -\frac{F_1}{g_A}, \quad (2.42)$$

$$f_M = -\left(\frac{G_M^V}{g_A} + \sqrt{\frac{2}{3}} f_1 \eta_4 \right), \quad (2.43)$$

$$f_c^5 = 1 + \frac{2}{\sqrt{3}}\eta_3, \quad (2.44)$$

$$f_T = -\frac{g_T}{g_A}, \quad (2.45)$$

where the isospin indices “ (\pm) ” for the single nucleon form factors are suppressed.

The Cartesian form of the T-odd response functions $R_7^{(2)}, \dots, R_{12}^{(2)}$ are

$$\begin{aligned} R_7^{(2)} = & \frac{\epsilon_e}{2M} \left(\mp Im f_c^5 + Im f_M - Im f_T \right) \\ & + \frac{\epsilon_e \Delta}{2M^2} \left(\frac{1}{7} \sqrt{\frac{7}{15}} Im f_1 \eta_6 \pm \frac{\sqrt{2}}{10} Im \eta_8 \pm \frac{1}{30} \sqrt{\frac{10}{7}} Im \eta_9 \pm \frac{4}{15} \frac{1}{\sqrt{7}} Im \eta_{10} \right) \\ & + \frac{\epsilon_e^2}{2M^2} \left(-\frac{2}{7} \sqrt{\frac{7}{15}} Im f_1 \eta_6 \mp \frac{\sqrt{2}}{5} Im \eta_8 \mp \frac{1}{15} \sqrt{\frac{10}{7}} Im \eta_9 \pm \frac{8}{15\sqrt{7}} Im \eta_{10} \right), \end{aligned} \quad (2.46)$$

$$\begin{aligned} R_8^{(2)} = & -\frac{\Delta - \epsilon_e}{2M} \left(\pm Im f_c^5 + Im f_M + Im f_T \right) \\ & + \frac{\Delta^2}{M^2} \left(\frac{1}{14} \sqrt{\frac{7}{15}} Im f_1 \eta_6 \pm \frac{\sqrt{2}}{10} Im \eta_8 \mp \frac{1}{30} \sqrt{\frac{10}{7}} Im \eta_9 \pm \frac{4}{15\sqrt{7}} Im \eta_{10} \right) \\ & - \frac{\Delta \epsilon_e}{M^2} \left(\frac{5}{14} \sqrt{\frac{7}{15}} Im f_1 \eta_6 \pm \frac{3\sqrt{2}}{10} Im \eta_8 \mp \frac{1}{30} \sqrt{\frac{10}{7}} Im \eta_9 \pm \frac{4}{5\sqrt{7}} Im \eta_{10} \right) \\ & - \frac{\epsilon_e^2}{M^2} \left(-\frac{2}{7} \sqrt{\frac{7}{15}} Im f_1 \eta_6 \mp \frac{\sqrt{2}}{5} Im \eta_8 \mp \frac{1}{15} \sqrt{\frac{10}{7}} Im \eta_9 \mp \frac{8}{15\sqrt{7}} Im \eta_{10} \right), \end{aligned} \quad (2.47)$$

$$\begin{aligned} R_9^{(2)} = & -\frac{\epsilon_e \Delta}{M^2} \sin(\delta_2 - \delta_0) \left(\frac{1}{2} \sqrt{\frac{2}{3}} Re f_1 \eta_6(2^+, 0^+) \pm \frac{1}{15} Re \eta_9(2^+, 0^+) \right) \\ & + \frac{2\epsilon_e^2}{M^2} \sin(\delta_2 - \delta_0) \left(\frac{2}{5} \sqrt{\frac{2}{3}} Re f_1 \eta_6(2^+, 0^+) \pm \frac{1}{15} Re \eta_9(2^+, 0^+) \right), \end{aligned} \quad (2.48)$$

$$R_{10}^{(2)} = -\frac{\Delta^2}{M^2} \sin(\delta_2 - \delta_0) \left(\frac{3}{10} \sqrt{\frac{2}{3}} Re f_1 \eta_6(2^+, 0^+) \pm \frac{1}{15} Re \eta_9(2^+, 0^+) \right)$$

$$\begin{aligned}
& + \frac{\Delta\epsilon_e}{M^2} \sin(\delta_2 - \delta_0) \left(\frac{11}{10} \sqrt{\frac{2}{3}} \text{Re} f_1 \eta_6(2^+, 0^+) \pm \frac{1}{5} \text{Re} \eta_9(2^+, 0^+) \right) \\
& - \frac{2\epsilon_e^2}{M^2} \sin(\delta_2 - \delta_0) \left(\frac{2}{5} \sqrt{\frac{2}{3}} \text{Re} f_1 \eta_6(2^+, 0^+) \pm \frac{1}{15} \text{Re} \eta_9(2^+, 0^+) \right), \quad (2.49)
\end{aligned}$$

$$\begin{aligned}
R_{11}^{(2)} &= \frac{\epsilon_e \Delta}{M^2} \sin(\delta_2 - \delta_4) \left(\frac{2}{5} \sqrt{\frac{3}{7}} \text{Re} f_1 \eta_6(2^+, 4^+) \pm \frac{2}{15} \sqrt{\frac{2}{7}} \text{Re} \eta_9(2^+, 4^+) \right) \\
& - \frac{2\epsilon_e^2}{M^2} \sin(\delta_2 - \delta_4) \left(\frac{4}{15} \sqrt{\frac{3}{7}} \text{Re} f_1 \eta_6(2^+, 4^+) \pm \frac{2}{15} \sqrt{\frac{2}{7}} \text{Re} \eta_9(2^+, 4^+) \right), \quad (2.50)
\end{aligned}$$

$$\begin{aligned}
R_{12}^{(2)} &= \frac{\Delta^2}{M^2} \sin(\delta_2 - \delta_4) \left(\frac{2}{15} \sqrt{\frac{3}{7}} \text{Re} f_1 \eta_6(2^+, 4^+) \pm \frac{2}{15} \sqrt{\frac{2}{7}} \text{Re} \eta_9(2^+, 4^+) \right) \\
& - \frac{\Delta\epsilon_e}{M^2} \sin(\delta_2 - \delta_4) \left(\frac{2}{3} \sqrt{\frac{3}{7}} \text{Re} f_1 \eta_6(2^+, 4^+) \pm \frac{2}{15} \sqrt{\frac{2}{7}} \text{Re} \eta_9(2^+, 4^+) \right) \\
& + \frac{2\epsilon_e^2}{M^2} \sin(\delta_2 - \delta_4) \left(\frac{4}{15} \sqrt{\frac{3}{7}} \text{Re} f_1 \eta_6(2^+, 4^+) \pm \frac{2}{15} \sqrt{\frac{2}{7}} \text{Re} \eta_9(2^+, 4^+) \right), \quad (2.51)
\end{aligned}$$

where

$$\eta_6(2^+, J^+) = M^2 \frac{\sum_{i=1}^A \langle J^+ || \tau^{(\pm)} r^2 Y_2 || 2^+ \rangle}{\sum_{i=1}^A \langle 2^+ || \tau^{(\pm)} \mathcal{Y}_{10} \cdot \sigma || 2^+ \rangle}, \quad (2.52)$$

$$\eta_9(2^+, J^+) = M^2 \frac{\sum_{i=1}^A \langle J^+ || \tau^{(\pm)} r^2 \mathcal{Y}_{22} \cdot \sigma || 2^+ \rangle}{\sum_{i=1}^A \langle 2^+ || \tau^{(\pm)} \mathcal{Y}_{10} \cdot \sigma || 2^+ \rangle}, \quad (2.53)$$

$$(2.54)$$

with $J^P = 0^+, 4^+$. Terms in Eqs. 2.48–2.51 that are proportional to $\cos(\dots)$ have been dropped, since they are much smaller than those terms propor-

tional to $\sin(\dots)$ due to the fact they are multiplied by T-odd hadronic response functions which are expected to be much smaller than the corresponding T-even components even if the Hamiltonian of the system is not invariant under time reversal. The Fermi matrix element A_1 is assumed to be zero.

d) Shell model calculation of the matrix elements of the static multipole operators in the A=8 system

The intermediate states of the β -decay processes are ${}^8\text{Be}^*$ resonances. Here, we shall consider a shell model computation of the reduced matrix elements of various static multipole operators that are used in the next section where the decay of these shell model states are discussed and where a comparison with experiments is made.

We shall restrict ourselves to a $1p$ shell model space. Using the effective interaction developed by Cohen and Kurath [21](2BME 8-16), the shell model Hamiltonian can be diagonalized [22]. The $1p$ configuration mixing shell model calculation with the Cohen-Kurath potential (2BME 8-16) is used to calculate the $J^P = 2^+$ levels of ${}^8\text{Be}^*$ (see Fig. 2); these are stable shell model states that lie at $E=3.55(0)$, $13.37(0)$, $16.19(1)$, $17.45(0)$,... (the numbers in parentheses are the isospin for the corresponding states and the unit is in MeV). These theoretically determined energy levels lie reasonably close to the experimental resonant state energies given in Fig. 3. The shell model state at $E=3.55$ corresponds to the $E=3.04$ state and that at $E=16.19$ corresponds to the $E=16.63$ state in Fig. 2. The shell model state at $E=13.37$ possesses similar one particle parentage coefficient in its wave function [23] as the shell model state located at $E=16.92$, so although it is relatively far from the state at $E=16.92$, it should be identified with that state. The $E=17.45$ state corresponds to some higher excitation state in the $J^P = 2^+$ state series; it is not considered in this paper.

The doubly reduced matrix element of an one body transition operator

has the following form [19, 24]

$$\langle J_1; T_1 :: O_{J,T}^{(1)} :: J_2; T_2 \rangle = \sum_{aa'} \psi_{J,T}(a'a) \langle a' :: O_{J,T}^{(1)} :: a \rangle, \quad (2.55)$$

where a, a' enumerate single nucleon shell model states. In the $1p$ shell model subspace, $a = j = (1/2, 3/2)$, with j the single nucleon total angular momentum. The density matrix $\psi_{J,T}(a', a)$ is defined in Ref. [24]; it depends on the shell model Hamiltonian in the $1p$ shell subspace. The reduced matrix elements of various static multipole operators defined in Table 3 are

$$A_i = \pm \frac{1}{\sqrt{3}} \text{Tr} \psi^T \mathbf{A}_i, \quad (2.56)$$

where “T” denotes transpose and ψ is the corresponding density matrix.

By using harmonic oscillator radial wave functions [24], the one nucleon matrix elements corresponding to those defined in Table 3 are

$$\mathbf{A}_1 = \frac{\langle I_T \rangle}{\sqrt{4\pi}} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 2 \end{pmatrix}, \quad (2.57)$$

$$(2.58)$$

$$\mathbf{A}_2 = \frac{\langle I_T \rangle}{\sqrt{4\pi}} \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{4}{\sqrt{3}} \\ \frac{-4}{\sqrt{3}} & 2\sqrt{\frac{5}{3}} \end{pmatrix}, \quad (2.59)$$

$$(2.60)$$

$$\mathbf{A}_3 = \frac{\langle I_T \rangle}{\sqrt{4\pi}} \begin{pmatrix} \frac{1}{\sqrt{2}} & -5 \\ -1 & -\sqrt{5} \end{pmatrix}, \quad (2.61)$$

$$(2.62)$$

$$\mathbf{A}_4 = \frac{\langle I_T \rangle}{\sqrt{4\pi}} \begin{pmatrix} -2 & \sqrt{2} \\ -\sqrt{2} & -\sqrt{10} \end{pmatrix}, \quad (2.63)$$

$$(2.64)$$

$$\mathbf{A}_5 = \frac{\langle I_T \rangle}{\sqrt{4\pi}} \begin{pmatrix} \frac{5}{\sqrt{2}} & 0 \\ 0 & 5 \end{pmatrix} (Mb)^2, \quad (2.65)$$

$$(2.66)$$

$$\mathbf{A}_6 = \frac{\langle I_T \rangle}{\sqrt{4\pi}} \begin{pmatrix} 0 & 5 \\ -5 & -5 \end{pmatrix} (Mb)^2, \quad (2.67)$$

$$(2.68)$$

$$\mathbf{A}_7 = \frac{\langle I_T \rangle}{\sqrt{4\pi}} \begin{pmatrix} -\frac{5}{2}\sqrt{\frac{2}{3}} & \frac{10}{\sqrt{3}} \\ -\frac{10}{\sqrt{3}} & 5\sqrt{\frac{5}{3}} \end{pmatrix} (Mb)^2, \quad (2.69)$$

$$(2.70)$$

$$\mathbf{A}_8 = \frac{\langle I_T \rangle}{\sqrt{4\pi}} \begin{pmatrix} -\frac{10}{\sqrt{3}} & -\frac{5}{2}\sqrt{\frac{2}{3}} \\ \frac{5}{2}\sqrt{\frac{2}{3}} & \sqrt{\frac{10}{3}} \end{pmatrix} (Mb)^2, \quad (2.71)$$

$$(2.72)$$

$$\mathbf{A}_9 = \frac{\langle I_T \rangle}{\sqrt{4\pi}} \begin{pmatrix} 0 & -5\sqrt{\frac{3}{2}} \\ -5\sqrt{\frac{3}{2}} & 0 \end{pmatrix} (Mb)^2, \quad (2.73)$$

$$(2.74)$$

$$\mathbf{A}_{10} = \frac{\langle I_T \rangle}{\sqrt{4\pi}} \begin{pmatrix} 0 & 0 \\ 0 & -15\sqrt{\frac{7}{15}} \end{pmatrix} (Mb)^2, \quad (2.75)$$

where the matrices \mathbf{A}_i ($i=1, \dots, 10$) act on the $j = 1/2$ and $j = 3/2$ spaces with the 11 (first) element of the matrices corresponding to the diagonal matrix element for $j = 1/2$. The single nucleon isospin factor $\langle I_T \rangle = \sqrt{3}$. b is the oscillator parameter for the shell model orbits.

3 Semi-phenomenological R-matrix treatment for the $A = 8$ system

a) R-Matrix theory for β -decay processes

Following Appendix D, the T-matrix for the β -decay processes is found to be

$$T_{fi} = \sum \frac{\Gamma_{2\alpha,n}(E_{2\alpha}) \langle n; e\nu | H_w | \phi_i \rangle}{E_{2\alpha} - E_n - R_n(E_{2\alpha})}. \quad (3.1)$$

$R_n(E)$ is complex, namely

$$R_n(E) = D_n(E) - iI_n(E), \quad (3.2)$$

where the real functions $D_n(E)$ and $I_n(E)$ are the level shift and the half width^{3.1} of the n th level, respectively. $\Gamma_{2\alpha,n}(E)$ is the vertex function that connects the n th level resonance to the two α states. The result given in Eq. 3.1 is formally exact, provided $|n\rangle$ is a complete set in the Hilbert space of interest. All effects of ΔV are represented by the functions $R_n(E)$ and $\Gamma_n(E)$. We shall not use a complete set of $\langle n|$; we only need a few of the most important shell model states, as shown in Appendix D. In addition, only the (strong and electromagnetic) interaction effects at short distances ($r < 5fm$) are included in $R_n(E)$ and $\Gamma_n(E)$. The residual effects are written as

$$T_{fi} = \sum_{shell\ model} (\dots) + \delta T_{fi}, \quad (3.3)$$

where residual part δT_{fi} will be specified in the following.

The imaginary part of $R_n(E)$ is related to the magnitude of $\Gamma_n(E)$ through the optical theorem [26]. In the particular energy range ($0 < E < 16$ MeV) considered here, the only open channel into which the shell model states $|n\rangle$ can decay is the 2α one. The optical theorem requires that

$$I_n(E) = \pi \rho(E) |\Gamma_{2\alpha,n}(E)|^2, \quad (3.4)$$

$$\rho(E) = \frac{1}{2} M_\alpha \sqrt{M_\alpha E}, \quad (3.5)$$

where we have assumed the center of mass (c.m.) frame for ${}^8\text{Be}^*$ and used a non-relativistic approximation

$$E = \frac{k^2}{M_\alpha}, \quad (3.6)$$

with k the relative momentum between the two α particles.

At low energies, the R-matrix parameters $I_n(E)$ and $\Gamma_{2\alpha,n}(E)$ are independent of the details of the interaction potential ΔV . Their asymptotic behavior can be derived from Eqs. D.7, and D.8. Namely, if ΔV has a short range (the long range Coulomb interaction between two α particles is treated

^{3.1}The width of the n th level is defined as the value of $I_n(E)$ at the energy where $E - E_n - D_n(E) = 0$.

separately in this study) and is less singular than^{3.2} $1/r^{2-\epsilon}$ ($\epsilon \ll 1$) as $r \rightarrow 0$, then

$$|2\alpha\rangle \stackrel{|\mathbf{k}| \rightarrow 0}{\sim} (a|\mathbf{k}|)^J, \quad (3.7)$$

$$|\psi_n\rangle \stackrel{|\mathbf{k}| \rightarrow 0}{\sim} \text{constant}, \quad (3.8)$$

where a is some scale factor of dimension length that is roughly the size of ΔV and J is the total angular momentum of the state with two asymptotic α particles as $r \rightarrow \infty$. By combining Eqs. 3.7, 3.8 and D.7, one gets

$$\Gamma_{2\alpha,n}(E) \stackrel{E \rightarrow 0}{\sim} E^{\frac{J}{2}}, \quad (3.9)$$

where use has been made of the non-relativistic spectrum for the α particles Eq. 3.6.

From Eq. 3.4, we get

$$I_n(E) \stackrel{E \rightarrow 0}{\sim} E^{J+\frac{1}{2}}. \quad (3.10)$$

At low energies, $D_n(E)$ and $I_n(E)$ are expected to be smooth functions of E . They can be expanded in a power series of E with the expansion coefficients treated as parameters. These parameters are determined from experimental data for α - α scattering phase shifts and allowed β -decay rates in the $A = 8$ system.

We shall require that $D_n^J(E)$ is analytic in the energy region of interest. The denominator $E - E_n^J - D_n^J(E)$ can therefore be written as

$$E - E_n^J - D_n^J(E) = E - E_n^{0,J} - a(E - E_n^{0,J})^2 - b(E - E_n^{0,J})^3 + \dots \quad (3.11)$$

with $E_n^{0,J}$ the experimental resonant energy of the n th level of total angular momentum J . Terms of higher power in $E - E_n^{0,J}$ represented by “...” are

^{3.2}Albeit the definition of r can be ambiguous if the α particle is composite and the overlap between the two α particles is strong, the singularity of the effective ΔV can be smoother than the one extracted from ΔV at larger distance where the overlap between these particles are negligible due to the compositeness of the particles. Therefore the statement based on point particle pairs given here is also expected to be true for composite particles.

expected to be unimportant around resonant energies. The parameters a and b are to be determined in the following.

Next, let's consider the vertex function. From the low energy behavior given by Eq. 3.9, $\Gamma_n^J(E)$ can be written as

$$\Gamma_n^J(E) = N_0 E^{\frac{J}{2}} (1 - cE + \dots), \quad (3.12)$$

with “...” representing higher order terms in powers of E that can be ignored at low energies, and N_0, c are coefficients. The half width function $I_n^J(E)$ is related to $\Gamma_n^J(E)$ through the optical theorem, Eq. 3.4. In the following, we shall express the magnitude of $\Gamma_n^J(E)$, in terms of $I_n^J(E)$ and parameterize $I_n^J(E)$ as

$$I_n^J(E) = I_n^{0,J} \sqrt{\frac{E}{E_n^{0,J}}} \left(\frac{E}{E_n^{0,J}} \right)^J \frac{(1 - cE)^2}{(1 - cE_n^{0,J})^2}, \quad (3.13)$$

where $I_n^{0,J}$, and c are coefficients to be determined. The vertex function $\Gamma_n^J(E)$ is assumed to be real function of E on the real E -axis.

b) α - α scattering

The R-matrix theory can also be applied to the scattering of two α particles. The collision of two α particles can be viewed as proceeding through various resonant levels of the $A = 8$ system, plus potential scattering between the two α particles at larger distances where the overlap between the two α particles is small. The rigorous relationship between the R-matrix parameters in the β -decay processes and the resonant scattering cross section of the α particles is indirect. Thus, a more explicit ansatz has to be made in order to relate the β -decay and the α - α scattering processes in a way that is convenient for the phenomenological applications.

We assume that the scattering phase shifts of the α particles due to interactions at short distances (the resonant scattering), $\delta_J^{(res)}(E)$, and at large distances (the potential scattering), $\delta_J^{(pot)}(E)$, are additive. The resonant part of the T-matrix for the collision of the two α -particles in a state of

total angular momentum J can be related to phase shifts generated by the resonant states and by the potential scattering as

$$t_J(E) = -\frac{2}{\pi M_\alpha k} e^{i\delta_J(E)} \sin \delta_J(E), \quad (3.14)$$

$$\delta_J(E) = \delta_J^{(res)}(E) + \delta_J^{(pot)}(E), \quad (3.15)$$

We shall assume further that the vertex function in the R-matrix does not contain any contribution from the potential scattering at large distances, and that the potential scattering is provided solely by the long range Coulomb interaction between the two α particles. The phase shift due to the potential scattering is expressed as

$$\tan \delta_J^{(pot)}(E) = -F_J(rk)/G_J(rk), \quad (3.16)$$

where $F_J(x)$ and $G_J(x)$ are the regular and irregular Coulomb functions [28] for the two α -particle system, and r is the distance between the two α particles where their relative radial wave functions vanish when $\Delta V = 0$. It follows that there are infinite number of discrete values of r that lead to the same potential phase shift given by Eq. 3.16; all of them are energy dependent. The energy dependence of r is expected to be smaller when it is close to the origin, especially when the interaction between the two α particles is stiff enough at small distances that it can be treated as a hard core. The energy dependence of r is thus expected to be minimized by choosing it fairly close to the origin ^{3.3}.

The T-matrix corresponding to the interactions that are responsible for the resonant states (see Fig. 5) in the $A = 8$ system can be expressed in

^{3.3}Since we can extrapolate the Coulomb wave function given by a combination of F_J and G_J at large distances all the way to points close to the origin where the strong interaction dominates, the value of r is not necessarily restricted to the region where the Coulomb interaction is most important. In fact, any value of r that gives the same energy dependence for the phase shifts is acceptable. The meaning of r as a measure of the range of the potential is meaningful only if the potential indeed contains a stiff piece (e.g., hard core). In this case, a value of r that is located near the hard core should be almost energy independent. Given the energy dependence of $\delta_J^{(pot)}$, it must be emphasized that, in the region where the strong interaction is important, the zeros of the true wave function of the system are not located at the ones satisfying Eq. 3.16

terms of the R-matrix parameters, namely

$$\begin{aligned} t_J^{(res)}(E) &= -\frac{2}{\pi M_\alpha k} e^{i\delta_J^{(res)}(E)} \sin \delta_J^{(res)}(E), \\ &= \sum \frac{|g_{2\alpha,n}^J(E)|^2}{E - E_n^J - D_n^J(E) + iw_n^J(E)}, \end{aligned} \quad (3.17)$$

where it is assumed that the level shift function $D_n^J(E)$ is the same as that in the R-matrix. The relationship between $g_n^J(E)$ and $w_n^J(E)$ can be obtained by unitarity considerations of the α - α T-matrix; on the other hand, the relationship between $w_n^J(E)$ and $I_n^J(E)$ is not obvious. $\Gamma_n^J(E)$ is the half width of the corresponding resonant level or the inverse life time of that level. The decay of the level to two α particles can be pictured as that of a prepared state, namely as the eigenstate of the shell model Hamiltonian H_{shell} , tunneling through the potential “barrier” ΔV . From the time reversal invariance consideration, it follows that in case of resonant α - α scattering, the time needed for the α particles to tunnel into the potential “barrier” ΔV to form a well prepared resonant state (corresponding to a shell model eigenstate) is the same as the time it takes the same state to tunnel through the potential “barrier” ΔV on its way out. Thus the time delay due to the α particle staying in a resonant state in α - α scattering is twice as long as the time for the corresponding shell model eigenstate to decay near a resonant energy of that state. This qualitative argument is formally, albeit not explicitly, derived in Ref. [26]. From these considerations, it is assumed that

$$w_n^J(E) = \frac{1}{2} I_n^J(E). \quad (3.18)$$

As will be shown in the following, this relation leads to an excellent synthesis between the β -decay α spectra and α - α scattering within the R-matrix formalism and without the need of any “intruder” states.

Since a two spin zero particle state with definite total angular momentum J has only one phase shift at a given energy, the phase for each of the terms in the above sum has to be the same, which allows us to write Eq. 3.17 as

$$\tan \delta_J^{(res)}(E) = - \sum_n \frac{w_n^J(E)}{E - E_n^J - D_n^J(E)}. \quad (3.19)$$

Together with Eq. 3.16, this equation yields

$$\delta_J(E) = -\tan^{-1} \sum_n \frac{w_n^J(E)}{E - E_n^J - D_n^J(E)} - \tan^{-1} \frac{F_J(rk)}{G_J(rk)}. \quad (3.20)$$

Before going on to further specifications of $I_n^J(E)$ and $D_n^J(E)$, it is useful to write the T-matrix for the α - α scattering in the total angular momentum J state as

$$-\frac{\pi}{2} M_\alpha k t_J(E) = e^{i\delta_J^{(res)}(E) + i\delta_J^{(pot)}(E)} \left[\sin\delta_J^{(res)}(E) \cos\delta_J^{(pot)}(E) + \cos\delta_J^{(res)}(E) \sin\delta_J^{(pot)}(E) \right]. \quad (3.21)$$

The parameter “r” in the Coulomb functions $F_J(rk)$ and $G_J(rk)$ is also expected to be energy dependent. The form of its energy dependence is not easy to determine from known general properties of the system. So we shall take the following trial form

$$r(E) = r_0(1 - zE^2), \quad (3.22)$$

with “z” a parameter to be determined. It leads to a good fit to the experimental data.

$J^P = 0^+$ states:

The $J^P = 0^+$ states are shown in Fig. 3. The ground state of ${}^8\text{Be}^*$ is in the $J^P = 0^+$ series and is rather narrow. The next two states in this series are broad and have resonant energies at $E = 20.2\text{MeV}$ and $E = 27.49\text{MeV}$ respectively. Since these states are well separated from the ground state, it is justifiable to include only the ground state in the R-matrix at low energies ($E^{cm} \leq 12\text{ MeV}$).

The best fit (least square fitting) to the experimental α - α phase shifts [29] in the $J^P = 0^+$ state is achieved by using the set of parameters shown in Table 4. The result of the fitting procedure is shown in Fig. 6. The value of the $J^P = 0^+$ phase shift at $E = 0.4\text{MeV}$ given in Ref. [29] is interpreted as $180^\circ \pm 0.5$ instead of $0^\circ \pm 0.5$ in order to take into account the fact that the 0^+ resonance is at about $E = 0.2\text{MeV}$, which is smaller than 0.4MeV .

$J^P = 2^+$ states:

The $J^P = 2^+$ states are our prime interest. Fig. 3 shows a series of experimental low lying states with $J^P = 2^+$. In the energy region considered here, we shall assume that only the lowest three states are important, namely the states with energy $E = 3.04$ MeV, 16.63 MeV and 16.92 MeV. Higher states are assumed to have negligible influence for ~ 12 MeV. The state at $E = 3.04$ MeV is an isosinglet state. The nearly degenerate doublet states at an energy near 16 MeV are mixtures of isosinglet and isotriplet. The latter component does not contribute to the α - α scattering to lowest order in the fine structure constant $\alpha = e^2/\hbar c$. The mixing angle between the isosinglet and the isotriplet is nearly 45° [30, 31]. This mixing angle will be determined by fitting the α particle spectra in the β -decay processes. They are not relevant to the phase shift analysis^{3,4}. In addition, the widths of these two states have very small influence on the α - α phase shift in the energy region considered in Fig. 7.

The best least square fit to the experimental α - α phase shifts [29] in the $J^P = 2^+$ state is achieved by using the set of parameters shown in Table 5, in which the widths of the two 16 MeV doublet states are not determined from the α - α scattering phase shift but rather from the α -spectra in the β -decay processes. The result of the fitting is shown in Fig. 7.

It can be noted that the value of r obtained in the $J^P = 2^+$ channel is smaller than two times the α particle size. This feature should not be regarded as a shortcoming of the procedure. As is pointed out in section 3.a, the value of r is not necessarily restricted to be outside of the region where the α - α overlap is unimportant. In our approach, r represents the location of a zero of the radial wave function before the decay potential ΔV is turned on. From this point of view, there are many equivalent r 's.

$J^P = 4^+$ state:

The $J^P = 4^+$ series of states are also shown in Fig. 3. The lowest state

^{3,4} This conclusion follows because the α - α phase shift expressed by the R-matrix contains no information about the resonance other than its width and the location. Without further assumptions, the isospin mixing of the resonant states is not easily determined by the phase shift analysis in the present approach.

is at 11.4 MeV. Higher states in the $J^P = 4^+$ series will be neglected in the following.

The best least square fit to the experimental [29] α - α phase shifts for $J^P = 4^+$ is achieved by using the set of parameters shown in Table 6. The result of the fitting procedure is shown in Fig. 8.

c) The α particle spectra in the β -decay processes of the A=8 system

The unpolarized differential β -decay rate follows from Eqs. 2.33 and 2.34, namely

$$\frac{dW}{dE_r} = \frac{2}{5\pi^2} G_F^2 \cos^2 \theta_c |g_A|^2 M_\alpha \sqrt{M_\alpha E_r} |A_2(E_r)|^2 D(E_r), \quad (3.23)$$

$$D(E_r) = \int_{m_e}^{E_0 - E_r} d\epsilon_e \epsilon_e \sqrt{\epsilon_e^2 - m_e^2} (E_0 - E_r - \epsilon_e)^2 F(Z, \epsilon_e) R_1^{(0)}(E_r, \epsilon_e), \quad (3.24)$$

$$F(Z, \epsilon_e) = \frac{2\pi\eta}{e^{2\pi\eta} - 1}, \quad (3.25)$$

$$\eta = \mp \frac{Z\alpha\epsilon_e}{\sqrt{\epsilon_e^2 - m_e^2}}, \quad (3.26)$$

with $Z = 4$ and $A_2(E_r)$ the energy dependent matrix element obtained from the shell model amplitude Eq. 2.35 using the R-matrix formalism (see the following).

From Eq. 3.1, it follows that the matrix element between the initial ${}^8\text{Li}$ or ${}^8\text{B}$ state and the final resonant state ${}^8\text{Be}^*$ that eventually decays to two α particles can be related to a set of shell model matrix elements between the same initial state and a discrete set of shell model ${}^8\text{Be}^*$ states through

$$\langle J_f || O_J || J_i \rangle = e^{i\delta_{J_f}(E)} \sum \frac{\Gamma_n^{J_f}(E_r) \langle J_f; n || O_J || J_i \rangle}{\sqrt{[E_r - E_n^{J_f} - D_n^{J_f}(E_r)]^2 + I_n^{J_f}(E_r)^2}}, \quad (3.27)$$

where $\langle J_f || O_J || J_i \rangle$ denotes any reduced matrix elements, and the index “n” enumerates different shell model states with the same parity and spin (total

angular momentum) J_f . The common phase factor for each of the terms in the summation is written as a multiplicative factor of the whole expression on the r.h.s. of Eq. 3.27 with the vertex function $\Gamma_{J_f}^n(E)$ real. This expression is, in principle, exact provided a complete set of shell model states is used and the energy dependent vertex function $\Gamma_n^J(E)$, level shift $D_n^J(E)$ and half width $I_n^J(E)$ are calculated from Eqs. D.6, D.7 and D.8. Due to the lack of knowledge of the decay potential ΔV and the complexity of the problem, a microscopic calculation of $\Gamma_n^J(E)$, $D_n^J(E)$ and $I_n^J(E)$ is beyond the scope of this work. Instead, we shall use the same parameterization of the R-matrix as in the α - α scattering case. Consistent with the treatment of the α - α scattering phase shift, only the 2^+ shell model states at $E=3.55$ MeV, $E=16.19$ MeV and $E=13.37$ MeV are used to evaluate the shell model transition amplitude. The energies of these states are taken from the fit of the α - α phase shifts. (This is reasonable, since it is expected that there are level shifts for the shell model states after switching on the decay potential ΔV). The shell model state lying at $E=17.45$ MeV is not included in the sum of Eq. 3.27. Two additional parameters in the β -decay processes that do not appear in the description of α - α scattering processes are needed. The first parameter is the isospin mixing angle between the isospin doublet members at $E \sim 16$ MeV. We shall define the mixing angle as

$$|E = 16.63\rangle = \cos\phi |T = 1\rangle + \sin\phi |T = 0\rangle, \quad (3.28)$$

$$|E = 16.92\rangle = -\sin\phi |T = 1\rangle + \cos\phi |T = 0\rangle, \quad (3.29)$$

where $|T = 0\rangle$ and $|T = 1\rangle$ denote the isosinglet and isotriplet shell model states respectively. Since only the $T=0$ components within each state of the doublet can decay into two α particles, the contribution of the $T=1$ part of each state is not included in the R-matrix. In addition, there are relative signs between $\Gamma^{J=2^+}(3.04)$, $\Gamma^{J=2^+}(16.63)$ and $\Gamma^{J=2^+}(16.92)$ that cannot be fixed from the sum rule relation Eqs. 3.4 and 3.5. They have to be determined in the fitting. The second parameter is related to the inclusion of the potential scattering in the R-matrix formalism for the β -decay processes consistent with the similar development for the α - α scattering. This parameter is necessary, since in the fitting of the α - α scattering phase shift, it is natural to

separate the potential scattering between the two α particles at long distances and the confining strong and Coulomb resonant interaction at short distances (characterized by the parameter r). Similar to Eq. 3.21 we shall write the T-matrix in the presence of the potential scattering between the α particles as

$$T(E) = e^{i\delta_J^{(pot)}(E)} \left[\cos\delta_J^{(pot)}(E) T_0(E) + \sin\delta_J^{(pot)}(E) \tilde{T}_0(E) \right]. \quad (3.30)$$

$T_0(E)$ has the form of Eq. 3.1 but with the full vertex function $\Gamma_n^J(E)$, the level shift function $D_n^J(E)$, and the half width function $I_n^J(E)$ replaced by the ones obtained from the phase shift fitting procedure outlined in the previous section. $\tilde{T}_0(E)$ is given by

$$\tilde{T}_0(E) = C \sum \frac{\Gamma_n^J(E)}{I_n^J(E)} \frac{E - E_n^J - D_n^J(E)}{E - E_n^J - R_n^J(E)} \langle n; e\nu | H_w | \phi_i \rangle, \quad (3.31)$$

with constant C a parameter to be determined. The magnitude of C is a rough measure for the direct weak transition from the initial ${}^8\text{Li}$ or ${}^8\text{B}$ to the two α scattering states.

The two new parameters C and ϕ , along with the widths of the isodoublet states are adjusted so that a fit to the α particle spectra for the β^+/β^- decay in $A = 8$ system is achieved. The resulting values for C and ϕ are

$$\phi = 51.8^\circ, \quad (3.32)$$

$$C = 5.7 \times 10^{-3}. \quad (3.33)$$

The widths of the doublet state are

$$w_0(16.63) = 0.206 \text{ MeV}, \quad (3.34)$$

$$w_0(16.92) = 0.303 \text{ MeV}. \quad (3.35)$$

In addition, relative to $\Gamma^{J=2^+}(3.04)$, the signs of $\Gamma^{J=2^+}(16.63)$ and $\Gamma^{J=2^+}(16.92)$ are negative. Before ending this subsection, let's compare the isospin mixing angle obtained here with the one obtained in Ref. [31], namely $\phi = 50.3^\circ$. Thus, the value for the isospin mixing angle obtained here agrees well with that of Ref. [31].

The results of the theoretical calculation, with the nucleon weak form factors given in Table 7, are compared to the experimental [32] data in Figs. 9 and 10. The dashed curve represents a calculation without the final state potential scattering, namely $\delta^{(pot)} = 0$ and the solid curve is the full calculation. Figs. 11 and 12 are linear scale curves corresponding to Figs. 9 and 10.

In obtaining the excellent fit to the α particle spectra shown in Figs. 9 and 10, an upward energy shift of 0.17 MeV has been made to the theoretical expression Eq. 3.23. It is as if only the lowest resonant energy $J^P = 2^+$ state were shifted upward by roughly 0.17 MeV. This shift spoils the agreement between theory and the experiment both in the shape of the α spectra and the phase shift. Furthermore it is much too large to be attributable to an experimental energy calibration [33]. We do not understand its origin, only its necessity. In Warburton's fit to the α - α scattering phase shift and the α particle spectra of the β -decay processes of the $A=8$ system, a shift in the resonant energy of order 0.07 MeV was also required [34]. Henceforth we shall use the R-matrix development without this shift.

It may be worth mentioning that the Barker and Warburton R-matrix treatment, which is different from the one used here, obtains an equally good (if not better) fit to the α -spectra [34] without the need of low lying "intruder" states if the value of R_c is chosen to be around 4.5 fm. Their R-matrix formalism does not, however, allow a straightforward and consistent extension to the forbidden transition matrix elements. On the other hand, in the R-matrix developed here, the allowed and forbidden transition amplitudes are treated on an equal footing, so that the study of PTRIV phenomenae, which are intrinsically due to forbidden processes, can be studied with the R-matrix determined in the previous sections. This kind of "predictive power" is the main reason that we adopted the present approach.

4 PTRIV due to the final state interactions between two final α particles

a) Energy dependence of the PTRIV response functions

Eqs. 2.48–2.51 show that the dependence of R_1^{PTRIV} and R_2^{PTRIV} on the energy of the charged lepton energy ϵ_e is quadratic. However, their dependence on the two α particle relative energy is not trivial. First there are strong E_r dependence of $\eta_6(2^+, 0^+)$, $\eta_9(2^+, 0^+)$, $\eta_6(2^+, 4^+)$ and $\eta_9(2^+, 4^+)$. Figs. 13–16 give the energy dependence of these quantities. Functions $\eta_6(2^+, 0^+)$ and $\eta_9(2^+, 0^+)$ are dominated by the $J^P = 0^+$ resonance located at $E = 0$; they have a zero near $E \sim 3$ MeV. $\eta_6(2^+, 4^+)$ and $\eta_9(2^+, 4^+)$ are dominated by the $J^P = 4^+$ resonance located at $E \sim 11$ MeV; they do not possess any zero in the energy range 2 – 5 MeV. Also, they depend on factors $\sin(\delta_2 - \delta_0)$ and $\sin(\delta_2 - \delta_4)$, which originate from the phase factors in the wave functions (see Eqs. 2.6 and 3.30). There is an energy near $E \sim 3$ MeV where the $J^P = 0^+$ phase shift, δ_0 , is equal to the $J^P = 2^+$ state phase shift as can be seen by a comparison of Fig. 6 and Fig. 7. Therefore, there is a zero in $\sin(\delta_2 - \delta_0)$. However, $\sin(\delta_2 - \delta_4)$, does not have a zero in this energy range.

In a test of genuine time reversal invariance, which is associated with the time reversal invariance of the underlying Lagrangian, experiments should be designed to select events in which the PTRIV contributions are small. In Figs. 17 and 18 the value of R_1^{PTRIV} and R_2^{PTRIV} are plotted against E_r and ϵ_e for the β^+ decay of A=8 system. It can be seen that both of R_1^{PTRIV} and R_2^{PTRIV} pass through zero for several combinations of E_r and ϵ_e , along a line located at $E_r \sim 3$ MeV. R_1^{PTRIV} is small at low e^\pm energies (ϵ_e) and increases with ϵ_e ; it changes more quickly with E_r . R_2^{PTRIV} changes less quickly at $\epsilon_e \approx 5.5$ MeV than at other values of ϵ_e and possesses zeros somewhere near $E_r \approx 3$ MeV and $\epsilon_e \leq 8$ MeV. Figs. 19 and 20 are 3-dimensional plots of the values of R_1^{PTRIV} and R_2^{PTRIV} against E_r and ϵ_e for the β^- decay in the A=8 system. The general features of R_1^{PTRIV} are the same as those for the β^+ decay. R_2^{PTRIV} is somewhat different from the corresponding β^+ decay due to the $v - a$ interference effects. There still are zeros around $E_r \approx 3$ MeV at different ϵ_e , but R_1^{PTRIV} turns negative at high values of E_r . It changes slower with E_r when $\epsilon_e \approx 5.5$ MeV compared with its value at other values

of ϵ_e in the energy region (of E_r) considered.

b) PTRIV in case the charged lepton energy is not measured

In case the charged lepton energy is not measured, an average over its energy distribution has to be made. From Eq. 3.24, it is natural to define the averaged PTRIV response function as

$$\bar{R}_i^{PTRIV}(E_r) = \frac{1}{N} \int_{m_e}^{E_0-E_r} d\epsilon_e \epsilon_e \sqrt{\epsilon_e^2 - m_e^2} (E_0 - E_r - \epsilon_e)^2 F(E_r, \epsilon_e) R_i^{PTRIV}(E_r, \epsilon_e), \quad (4.1)$$

$$N = \int_{m_e}^{E_0-E_r} d\epsilon_e \epsilon_e \sqrt{\epsilon_e^2 - m_e^2} (E_0 - E_r - \epsilon_e)^2 F(E_r, \epsilon_e), \quad (4.2)$$

with $i = 1, 2$.

The E_r dependence of the average PTRIV response functions are shown in Figs. 21 and 22.

5 e^\pm -nucleus Coulomb interaction and related PTRIV

The treatment of the Coulomb effects in the β -decay processes has a long history [35, 36, 37]. However the coordinate space version of the treatments are not easily adapted to the formalism developed in this study. We shall use a somewhat different approach.

Although it is not necessary to consider the full effects of the charged lepton-nucleus scattering for the purpose of this paper, we nevertheless treat them here. The particular aspect of the Coulomb scattering effects of the charged lepton studied in detail here is the Coulomb scattering induced PTRIV effects in the β -decay processes of the A=8 system. Similar effects were studied by Jackson, Treiman and Wyld for general allowed β -decay processes [38]. The Coulomb PTRIV effects were later related to e^\pm -Coulomb

phase shifts [39], similar to the relationships developed in the previous section. When we go beyond the non-relativistic limit, infinite series of partial waves are required (see later sections of this chapter). For our specific problem, we shall not start with the spherical basis to calculate the contribution of the e^\pm -nucleus Coulomb scattering to the PTRIV, but rather make use of the integral form of the Dirac equation for the charged leptons. These sets of integral equations are relativistic Lippmann-Schwinger equations, with the boundary conditions built-in from the beginning.

a) Modified weak interaction Hamiltonian due to the e^\pm -nucleus Coulomb scattering

When the Coulomb scattering of the e^\pm due to the presence of the charged nucleus is considered, \mathbf{q} can have a distribution of values associated with different \mathbf{k}'_e at the point of the e^\pm emission. If we take into account the Coulomb scattering, the weak Hamiltonian can be written as

$$H_W = \frac{G_F}{\sqrt{2}} \int \frac{d^3q}{(2\pi)^3} \langle \mathbf{k}_e, \mathbf{k}_\nu | j_\nu(\mathbf{q}) | 0 \rangle J^\mu(-\mathbf{q}), \quad (5.1)$$

where the helicity indices for the leptons have been suppressed.

The matrix elements of the leptonic weak current operator in the β -decay processes can be written as

$$\begin{aligned} \langle \mathbf{k}_e; \mathbf{k}_\nu | j_\mu(\mathbf{q}) | 0 \rangle &= \int d^3x e^{-i\mathbf{q}\cdot\mathbf{x}} \langle \mathbf{k}_e; \mathbf{k}_\nu | j_\mu(\mathbf{x}) | 0 \rangle \\ &= (2\pi)^3 \delta(\mathbf{k}'_e + \mathbf{k}_\nu + \mathbf{q}) \\ &\quad \begin{cases} \bar{\psi}_{e^-, \mathbf{k}_e}^{(-)}(\mathbf{k}'_e) \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}(\mathbf{k}_\nu) & \text{for } \beta^- \text{ decay} \\ \bar{u}_\nu(\mathbf{k}_\nu) \gamma_\mu (1 - \gamma^5) \psi_{e^+, \mathbf{k}_e}^{(+)}(\mathbf{k}'_e) & \text{for } \beta^+ \text{ decay} \end{cases} \end{aligned} \quad (5.2)$$

with $u_\nu(\mathbf{k})$ and $v_{\bar{\nu}}(\mathbf{k})$ massless neutrino and antineutrino spinors respectively. Eqs. 5.1 and 5.2 can be compared to earlier work on the Coulomb corrections to the allowed β -decay in the *elementary particle approach* [40, 41, 42]. The Fourier transformation of the e^\pm wave functions are given by

$$\psi_{e^-, \mathbf{k}_e}(\mathbf{k}'_e) = \int d^3x e^{-i\mathbf{k}'_e \cdot \mathbf{x}} \psi_{e^-, \mathbf{k}_e}(\mathbf{x}), \quad (5.3)$$

$$\psi_{e^+, \mathbf{k}_e}(\mathbf{k}'_e) = \int d^3x e^{i\mathbf{k}'_e \cdot \mathbf{x}} \psi_{e^+, \mathbf{k}_e}(\mathbf{x}). \quad (5.4)$$

b) e^\pm wave functions and their relativistic Lippmann–Schwinger equations

The wave functions for the e^\pm obey Dirac equations for relativistic charged fermions in Coulomb fields

$$(h_0 + V + \beta m_e) \psi_{e^-} = \epsilon_e \psi_{e^-}, \quad (5.5)$$

$$(h_0 - V + \beta m_e) \psi_{e^+} = \epsilon_e \psi_{e^+}, \quad (5.6)$$

where $V = -Z\alpha/|\mathbf{x}|$ in coordinate space and

$$h_0 = -i\alpha \cdot \nabla + \beta m_e, \quad (5.7)$$

with $\alpha^i = \gamma^0 \gamma^i$ and $\beta = \gamma^0$. h_0 is the free Dirac particle Hamiltonian. A very detailed study of the scattering state solutions to Eq. 5.5 or 5.6 can be found in the book by Greiner, Müller, and Rafelski [43].

For our purpose, it is more convenient to use a relativistic Lippmann–Schwinger equation for the e^\pm wave functions, namely

$$\psi_{e^-, \mathbf{k}_e}^{(\pm)}(\mathbf{x}) = u(\mathbf{k}_e) e^{i\mathbf{k}_e \cdot \mathbf{x}} + \int d^3x' \langle \mathbf{x} | \frac{1}{\epsilon_e - h_0 \pm i\epsilon} T_{e^-}^{(\pm)} | \mathbf{x}' \rangle u(\mathbf{k}_e) e^{i\mathbf{k}_e \cdot \mathbf{x}'}, \quad (5.8)$$

$$\psi_{e^+, \mathbf{k}_e}^{(\pm)}(\mathbf{x}) = v(\mathbf{k}_e) e^{-i\mathbf{k}_e \cdot \mathbf{x}} + \int d^3x' \langle \mathbf{x} | \frac{1}{-\epsilon_e - h_0 \pm i\epsilon} T_{e^+}^{(\pm)} | \mathbf{x}' \rangle v(\mathbf{k}_e) e^{-i\mathbf{k}_e \cdot \mathbf{x}'}, \quad (5.9)$$

where $\epsilon_e \geq 0$. The free Dirac Hamiltonian h_0 and the matrix elements $\langle \mathbf{x} | \dots | \mathbf{x}' \rangle$ are 4×4 matrices in Dirac space. In fact, Eq. 5.9 corresponds to the negative energy solution of Eq. 5.8, but in order to use field theoretical language, ϵ_e is restricted to be positive and equations for e^- and e^+ are written separately. Similar to the free Dirac spinor [44], the positron wave function can be obtained from the electron wave function by a charge conjugation and a change of sign of the Coulomb potential, namely

$$\psi_{e^+, \mathbf{k}_e}^{(\pm)}[V_{e+}] = C \gamma^0 \psi_{e^-, \mathbf{k}_e}^{(\mp)*}[V_{e+}], \quad (5.10)$$

where V_{e+} is the Coulomb potential experienced by a positron and $[V]$ denotes the potential used in solving the corresponding wave function. Due to the long range nature of the Coulomb potential, the Lippmann–Schwinger equation may not be well defined. This difficulty is overcome by introducing a screened Coulomb field, which is the case in any realistic experimental situation. The “pure” Coulomb case is realized in the limit of an infinite screening length. In most of the following formal discussions, we shall assume that the above limiting procedure is taken.

The Dirac spinors $u(\mathbf{k}_e)$ and $v(\mathbf{k}_e)$ represent a free electron and positron respectively; they satisfy

$$(\not{k} - m_e)u(\mathbf{k}) = 0, \quad (5.11)$$

$$(\not{k} + m_e)v(\mathbf{k}) = 0. \quad (5.12)$$

$T_{e^-/e^+}^{(\pm)}$ is the T-matrix of the e^\pm in the Coulomb field of the nucleus under study. $T_{e^-}^{(\pm)}$ and $T_{e^+}^{(\pm)}$ satisfy equations

$$T_{e^-}^{(\pm)} = V + V \frac{1}{\epsilon_e - h_0 \pm i\epsilon} T_{e^-}^{(\pm)}, \quad (5.13)$$

$$T_{e^+}^{(\pm)} = (-V) + (-V) \frac{1}{-\epsilon_e - h_0 \pm i\epsilon} T_{e^+}^{(\pm)}. \quad (5.14)$$

In momentum space, the Lippmann–Schwinger equations for an electron and a positron in a Coulomb potential become

$$\psi_{e^-, \mathbf{k}_e}^{(\pm)}(\mathbf{k}'_e) = \left[(2\pi)^3 \delta(\mathbf{k}_e - \mathbf{k}'_e) + \frac{1}{\epsilon_e - \alpha \cdot \mathbf{k}'_e - \beta m_e \pm i\epsilon} T_{e^-}^{(\pm)}(\mathbf{k}'_e, \mathbf{k}_e) \right] u(\mathbf{k}_e), \quad (5.15)$$

$$\psi_{e^+, \mathbf{k}_e}^{(\pm)}(\mathbf{k}'_e) = \left[(2\pi)^3 \delta(\mathbf{k}_e - \mathbf{k}'_e) + \frac{1}{-\epsilon_e + \alpha \cdot \mathbf{k}'_e - \beta m_e \pm i\epsilon} T_{e^+}^{(\pm)}(\mathbf{k}'_e, \mathbf{k}_e) \right] v(\mathbf{k}_e). \quad (5.16)$$

Define the pair of operators

$$O_{e^-}^{(\pm)}(\mathbf{k}'_e, \mathbf{k}_e) = \frac{1}{\epsilon_e - \alpha \cdot \mathbf{k}'_e - \beta m_e \pm i\epsilon} T_{e^-}^{(\pm)}(\mathbf{k}'_e, \mathbf{k}_e), \quad (5.17)$$

$$O_{e^+}^{(\pm)}(\mathbf{k}'_e, \mathbf{k}_e) = \frac{1}{-\epsilon_e + \alpha \cdot \mathbf{k}'_e - \beta m_e \pm i\epsilon} T_{e^+}^{(\pm)}(\mathbf{k}'_e, \mathbf{k}_e). \quad (5.18)$$

The conjugate operators are

$$\bar{O}_{e^-}^{(\pm)}(\mathbf{k}_e, \mathbf{k}'_e) = T_{e^-}^{(\mp)}(\mathbf{k}_e, \mathbf{k}'_e) \frac{1}{\epsilon_e - \alpha \cdot \mathbf{k}'_e - \beta m_e \mp i\epsilon}, \quad (5.19)$$

$$\bar{O}_{e^+}^{(\pm)}(\mathbf{k}_e, \mathbf{k}'_e) = T_{e^+}^{(\mp)}(\mathbf{k}_e, \mathbf{k}'_e) \frac{1}{-\epsilon_e + \alpha \cdot \mathbf{k}'_e - \beta m_e \mp i\epsilon}. \quad (5.20)$$

These operators will be used when the matrix elements of the leptonic weak current operators are studied.

c) The leptonic tensor

Using operators given by Eqs. 5.17–5.20 the leptonic tensor corresponding to the trace of the bilinear products of the leptonic weak current operators becomes

$$\begin{aligned} Tr j^\mu(\mathbf{q}) j^{\nu\dagger}(\mathbf{q}') &= (2\pi)^6 \delta(\mathbf{q} + \mathbf{k}_e + \mathbf{k}_\nu) \delta(\mathbf{q}' + \mathbf{k}_e + \mathbf{k}_\nu) L^{\mu\nu} \\ &\quad + (2\pi)^3 \delta(\mathbf{q} + \mathbf{k}_e + \mathbf{k}_\nu) A^{\mu\nu}(\mathbf{q}') + (2\pi)^3 \delta(\mathbf{q}' + \mathbf{k}_e + \mathbf{k}_\nu) B^{\mu\nu}(\mathbf{q}) \\ &\quad + C^{\mu\nu}(\mathbf{q}, \mathbf{q}'), \end{aligned} \quad (5.21)$$

where $L^{\mu\nu}$ is given by Eq. 2.4. $A^{\mu\nu}$, $B^{\mu\nu}$ and $C^{\mu\nu}$ are

For the β^- decay

$$A^{\mu\nu}(\mathbf{q}') = 2Tr O_{e^-}(-\mathbf{k}_\nu - \mathbf{q}', \mathbf{k}_e)(\not{k}_e + m_e)\gamma^\mu \not{k}_\nu \gamma^\nu (1 - \gamma^5), \quad (5.22)$$

$$B^{\mu\nu}(\mathbf{q}) = 2Tr(\not{k}_e + m_e)\bar{O}_{e^-}(\mathbf{k}_e, -\mathbf{k}_\nu - \mathbf{q})\gamma^\mu \not{k}_\nu \gamma^\nu (1 - \gamma^5), \quad (5.23)$$

$$C^{\mu\nu}(\mathbf{q}, \mathbf{q}') = 2Tr O_{e^-}(-\mathbf{k}_\nu - \mathbf{q}, \mathbf{k}_e)(\not{k}_e + m_e)\bar{O}_{e^-}(\mathbf{k}_e, -\mathbf{k}_\nu - \mathbf{q}')\gamma^\mu \not{k}_\nu \gamma^\nu (1 - \gamma^5). \quad (5.24)$$

For the β^+ decay

$$A^{\mu\nu}(\mathbf{q}') = 2Tr(\not{k}_e - m_e)\bar{O}_{e^+}(\mathbf{k}_e, -\mathbf{k}_\nu - \mathbf{q}')\gamma^\mu \not{k}_\nu \gamma^\nu (1 - \gamma^5), \quad (5.25)$$

$$B^{\mu\nu}(\mathbf{q}) = 2Tr O_{e^+}(-\mathbf{k}_\nu - \mathbf{q}, \mathbf{k}_e)(\not{k}_e - m_e)\gamma^\mu \not{k}_\nu \gamma^\nu (1 - \gamma^5), \quad (5.26)$$

$$C^{\mu\nu}(\mathbf{q}, \mathbf{q}') = 2Tr O_{e^+}(-\mathbf{k}_\nu - \mathbf{q}, \mathbf{k}_e)(\not{k}_e - m_e)\bar{O}_{e^+}(\mathbf{k}_e, -\mathbf{k}_\nu - \mathbf{q}')\gamma^\mu \not{k}_\nu \gamma^\nu (1 - \gamma^5). \quad (5.27)$$

d) On-Shell Coulomb T-matrix to all orders in the fine structure constant α

Rigorous expressions for $A^{\mu\nu}, \dots, C^{\mu\nu}$ involving the T-matrices (on-shell and off-shell) of the charged lepton in the hadronic Coulomb field are expected to be very complicated. Due to the smallness of the fine structure constant $\alpha = 1/137$, it is reasonable to adopt an approximation in which only terms to the first order in $Z\alpha$ are considered. This however can not be done straightforwardly by truncating the Lippmann-Schwinger equation, since the truncation leads to infrared divergences for many of the PTRIV observables. Even when a screened Coulomb potential is used, the value of the T-matrix in the forward direction cannot be reliably obtained using the lowest order one, which is real, when the screening length is much larger than the hadronic scale. In fact, an all order (in α) calculation yields a rapidly varying phase for the T-matrix in the forward direction that cancels the infrared divergences in the physical observables. This phase can be calculated analytically in the non-relativistic limit [27] (i.e., when the momentum of the charged lepton is much less than its mass). Away from the non-relativistic limit, the Coulomb T-matrix does not have a closed form. It is however possible to express it in terms of the Coulomb phase shifts obtained in Refs. [45, 46] and more recently in e.g. Ref. [43]. The first order approximation in $Z\alpha$ is made only at the final step of the calculation. For the special case of the β -decay processes of the A=8 system, the electron (positron) energy is much large than its rest mass. This allows us to use the T-matrix in its extreme relativistic limit, where helicity is conserved. In this limit, it is easy to show that the Dirac structure of the T-matrix is very simple. It can be written as

$$t = t_2 \otimes 1, \quad (5.28)$$

with “1” an unit 4×4 matrix. The matrix elements of t between states with definite helicities have the form

$$t^{++}(\mathbf{k}', \mathbf{k}) = \cos \frac{\theta}{2} t_2(\epsilon, \theta, \phi), \quad (5.29)$$

$$t^{--}(\mathbf{k}', \mathbf{k}) = t^{++}(\mathbf{k}', \mathbf{k}), \quad (5.30)$$

$$t^{+-}(\mathbf{k}', \mathbf{k}) \sim t^{-+}(\mathbf{k}', \mathbf{k}) = 0, \quad (5.31)$$

where “+” represents helicity “up” and “−” represents helicity “down” and (θ, ϕ) are the polar angles between \mathbf{k} and \mathbf{k}' . Quantity $t_2(\epsilon, \theta, \phi)$, which is independent of ϕ , is related to the Coulomb phase shifts through [47]

$$t_2(\epsilon_e, \theta, \phi) = \frac{1}{\epsilon_e |\mathbf{k}_e|} \sum_{\kappa=1}^{\infty} \left[e^{\pm i\delta_{\kappa}} \sin \delta_{\kappa} - e^{\pm i\delta_{-\kappa+1}} \sin \delta_{-\kappa+1} \right] P'_{\kappa-1}(\cos \theta), \quad (5.32)$$

with $\delta_{\pm\kappa}$ Coulomb phase shifts of the doublet states with total angular momentum $j = |\kappa| - \frac{1}{2}$ and with opposite parities, $P'_L(x) = dP_L/dx$ and $P_L(x)$ is the Legendre polynomial. Due to the helicity conservation of the T-matrix, the phase shifts of states with the same total angular momentum and with opposite parities are the same. Using the fact that $\delta_{\kappa} = \delta_{-\kappa}$, Eq. 5.32 can be written as

$$t_2(\epsilon_e, \theta) = \frac{1}{\epsilon_e |\mathbf{k}_e|} \sum_{\kappa=1}^{\infty} \left[e^{\pm i\delta_{\kappa}} \sin \delta_{\kappa} - e^{\pm i\delta_{\kappa-1}} \sin \delta_{\kappa-1} \right] P'_{\kappa-1}(\cos \theta). \quad (5.33)$$

PTRIV correlation observables depend on the moments of the real part of the on shell Coulomb T-matrix if a first order approximation in $Z\alpha$ is taken in the final results. These moments are defined as

$$a_n = \frac{|\mathbf{k}_e|^3}{Z\alpha\epsilon_e} \int_{-1}^1 dx x^n \text{Re}[t_2(\epsilon_e, x)], \quad (5.34)$$

with $x = \cos \theta$. Taking a first order approximation in $Z\alpha$ for the Coulomb phase shifts δ_{κ} , a_n can be expressed as

$$a_n = \sum_{\kappa=1}^{\infty} (1 + (-1)^{n+\kappa}) \Delta f_{\kappa} - n \sum_{\kappa=1}^n \int_{-1}^1 dx x^{n-1} P_{\kappa-1}(x) \Delta f_{\kappa}, \quad (5.35)$$

$$\Delta f_{\kappa \neq 1} = \frac{1}{2\kappa(\kappa-1)} + \frac{1}{2(\kappa-1)!} \left[(\kappa-1)^2(\kappa-2) - \kappa(\kappa-1) + 2\epsilon(2-\kappa) \right], \quad (5.36)$$

$$\Delta f_1 = -\frac{1}{2} + \epsilon, \quad (5.37)$$

where $\epsilon = 0.57721566\dots$ is the Euler–Mascheroni constant and the Coulomb phase shift obtained in Ref. [43] has been used. The first few useful moments are

$$a_0 = c^{(+)}, \quad (5.38)$$

$$a_1 = c^{(-)} + \left(\frac{1}{2} - \epsilon\right), \quad (5.39)$$

$$a_2 = c^{(+)} + 1, \quad (5.40)$$

$$a_3 = c^{(-)} + \left(\frac{4}{3} - \frac{8}{5}\epsilon\right), \quad (5.41)$$

with

$$c^{(+)} = 2 \sum_{\kappa=2,4,\dots}^{\infty} \Delta f_{\kappa}, \quad (5.42)$$

$$c^{(-)} = 2 \sum_{\kappa=1,3,\dots}^{\infty} \Delta f_{\kappa}. \quad (5.43)$$

The numerical value of $c^{(+)}$ and $c^{(-)}$ are given by $c^{(+)} = -9.942 \times 10^{-2}$ and $c^{(-)} = 9.942 \times 10^{-2}$. It can be easily seen that all these moments diverge if a first order (in $Z\alpha$) approximation for the Coulomb T–matrix of the e^{\pm} is used^{5.1} due to the singularity of the first order T at $x = 1$. The results of the Coulomb T–matrix for the e^{\pm} to all order in $Z\alpha$ suppresses the divergence at $x = 1$ and results in finite moments a_n . In addition, it can be shown that, except for some special combination of the moments, a_n contains contributions from states with all possible total angular momenta when the mass of the e^{\pm} is not much larger than its 3–momentum (i.e. away from the non–relativistic limit). However, in the extreme relativistic limit, the average effects of these states, represented by $c^{(+)}$ for even n and $c^{(-)}$ for odd n, are small and the sum of $c^{(+)}$ and $c^{(-)}$ seems to be vanishingly small in a numerical evaluation of the corresponding series.

^{5.1}Some combinations of the moments remains finite even in the first order approximation for the Coulomb T–matrix.

e) Dispersive part of the Coulomb corrected leptonic weak current

Only the dispersive parts of Eqs. 5.22–5.27 will be studied in this paper, since they alone contribute to the PTRIV signals. In addition, to the first order in $Z\alpha$, the dispersive parts of Eqs. 5.22–5.27 originate from the imaginary part of the free Dirac particle propagators

$$\frac{1}{\epsilon_e - h_0 \pm i\epsilon} = \frac{P}{\epsilon_e - h_0} \mp i\pi\delta(\epsilon_e - h_0), \quad (5.44)$$

$$\frac{1}{-\epsilon_e - h_0 \pm i\epsilon} = \frac{P}{-\epsilon_e - h_0} \mp i\pi\delta(-\epsilon_e - h_0), \quad (5.45)$$

where “P” denotes principal value.

For β^- decay

$$D_{\pm}^{\mu\nu}|_{disp}(\mathbf{q}, \mathbf{q}') = \frac{i}{\pi} |\mathbf{k}_e| \text{Re}[t_2^{e-}(\mathbf{k}_e, \mathbf{k}'_e)] \text{Tr}(\not{k}_e \gamma^0 \not{k}'_e \mp \not{k}'_e \gamma^0 \not{k}_e) \gamma^\mu \not{k}_\nu \gamma^\nu (1 - \gamma^5), \quad (5.46)$$

and

for β^+ decay

$$D_{\pm}^{\mu\nu}|_{disp}(\mathbf{q}, \mathbf{q}') = -\frac{i}{\pi} |\mathbf{k}_e| \text{Re}[t_2^{e+}(\mathbf{k}_e, \mathbf{k}'_e)] \text{Tr}(\not{k}_e \gamma^0 \not{k}'_e \mp \not{k}'_e \gamma^0 \not{k}_e) \gamma^\nu \not{k}_\nu \gamma^\mu (1 - \gamma^5), \quad (5.47)$$

where $D_{\pm}^{\mu\nu}$ is related to $A^{\mu\nu}$ and $B^{\mu\nu}$ given in Eqs. 5.22 and 5.23 or Eqs. 5.25 and 5.26 through

$$D_{\pm}^{\mu\nu}(\mathbf{q}, \mathbf{q}') = \frac{1}{(2\pi)^3} \int dk_e k_e^2 [A^{\mu\nu}(\mathbf{q}) \pm B^{\mu\nu}(\mathbf{q}')] \delta(\epsilon_e - \epsilon_k),, \quad (5.48)$$

with $\mathbf{q} = -\mathbf{k}'_e - \mathbf{k}_\nu$, $\mathbf{q}' = -\mathbf{k}''_e - \mathbf{k}_\nu$, $k_e = |\mathbf{k}'_e| = |\mathbf{k}''_e|$ and $\epsilon_k = \sqrt{k_e^2 + m_e^2}$.

The symmetric (with respect to $A^{\mu\nu}$ and $B^{\mu\nu}$) and dispersive parts of the final e^\pm -nucleus Coulomb scattering contribution to the leptonic tensor given by $D_{+}^{\mu\nu}(\mathbf{q}, \mathbf{q}')$ has opposite time reversal transformation properties to

the non-interacting part of $L^{\mu\nu}$ given in Table 4. Therefore the T-odd kinematic functions of rank 2 are given in Table 8, which can be compared to the ones given in Table 1. The antisymmetric (with respect to $A^{\mu\nu}$ and $B^{\mu\nu}$) and dispersive parts of the final e^\pm -nucleus Coulomb scattering contribution to the leptonic tensor given by $D_-^{\mu\nu}(\mathbf{q}, \mathbf{q}')$ has the same time reversal transformation properties as the non-interacting part of $L^{\mu\nu}$. Both of these kinematic functions contain an extra variable $\hat{\mathbf{k}}'_e$ which has to be integrated out.

f) PTRIV correlation observables

The dispersive contribution of the e^\pm -nucleus Coulomb scattering amplitude to the T-odd differential β -decay rate dW of the A=8 system can therefore be written as

$$dW = 2\pi\delta(E_i - E_f - \epsilon_e - \epsilon_\nu) \frac{1}{2} G_F^2 \cos^2 \theta_c \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 q'}{(2\pi)^3} Tr \left[J^{\mu\dagger}(-\mathbf{q}) J^\nu(-\mathbf{q}') \right] Tr \left[j_\mu^\dagger(\mathbf{q}) j_\nu(\mathbf{q}') \right]_{disp}. \quad (5.49)$$

By using the hadronic trace formula [18], and after some algebra and integration over $d\Omega_{\mathbf{k}'_e}$, it is found that in addition to Eqs. 2.46–2.51, there are other terms that contribute to dW^{PTRIV} . Both $D_+^{\mu\nu}$ and $D_-^{\mu\nu}$ contributes to the PTRIV correlation observables. The PTRIV observable in the β -decay processes of the A=8 system can be written as

$$dW^{PTRIV} = dW_0 T_2(\hat{\mathbf{k}}) : \left[\left(R_1^{PTRIV} + R_{em,1}^{PTRIV} \right) \hat{\mathbf{k}}_e \hat{\mathbf{k}}_e \times \hat{\mathbf{k}}_\nu + \left(R_2^{PTRIV} + R_{em,2}^{PTRIV} \right) \hat{\mathbf{k}}_\nu \hat{\mathbf{k}}_e \times \hat{\mathbf{k}}_\nu \right], \quad (5.50)$$

where the tensorial contraction between the second rank Cartesian tensor T_2 and two vectors by symbol “:” is defined by Eqs. 2.36 and 2.37. It is found that to order κ^2 in the hadronic response function are

$$R_{em,1}^{PTRIV} = \mp Z\alpha \left[\frac{3\epsilon_e}{8M} \beta_1 \left(\pm f_c + f_M + f_T + \frac{1}{15} \sqrt{\frac{15}{7}} \frac{\Delta}{M} f_1 \eta_6 \right) + \right.$$

$$\left(\frac{\epsilon_e^2}{M^2} \beta_2 - \frac{\epsilon_e(\Delta - \epsilon_e)}{M^2} \beta_3 \right) \times \left(-\frac{1}{30} \sqrt{\frac{15}{7}} f_1 \eta_6 \mp \frac{\sqrt{2}}{20} \eta_8 \mp \frac{1}{60} \sqrt{\frac{10}{7}} \eta_9 \mp \frac{2}{15} \frac{1}{\sqrt{7}} \eta_{10} \right) \Bigg], \quad (5.51)$$

$$\begin{aligned} R_{em,2}^{PTRIV} = & \mp Z\alpha \left[\frac{\epsilon_e(\Delta - \epsilon_e)}{2M^2} \beta_1 \left(\pm \frac{12}{5} \sqrt{\frac{5}{21}} f_1 \eta_6 - \frac{\sqrt{2}}{15} \eta_8 + \frac{1}{5} \sqrt{\frac{10}{7}} \eta_9 - \frac{4}{35} \frac{1}{\sqrt{7}} \eta_{10} \right) \right. \\ & + \left(\frac{\epsilon_e(\Delta - \epsilon_e)}{M^2} \beta_3 - \frac{(\Delta - \epsilon_e)^2}{M^2} \beta_4 \right) \times \\ & \left. \left(-\frac{1}{30} \sqrt{\frac{15}{7}} f_1 \eta_6 \mp \frac{\sqrt{2}}{20} \eta_8 \mp \frac{1}{60} \sqrt{\frac{10}{7}} \eta_9 \mp \frac{2}{15} \frac{1}{\sqrt{7}} \eta_{10} \right) \right], \quad (5.52) \end{aligned}$$

where the reduced response functions η_i are given in section 2, and

$$\beta_1 = a_2 - a_0 = 1, \quad (5.53)$$

$$\beta_2 = 3a_3 + 3a_2 - a_1 - a_0 = 6 - \frac{14}{5}\epsilon, \quad (5.54)$$

$$\beta_3 = 3a_2 + 2a_1 - a_0 = 4 - 2\epsilon, \quad (5.55)$$

$$\beta_4 = 2a_1 + 2a_0 = 1 - 2\epsilon. \quad (5.56)$$

In order to test whether or not the underlying Lagrangian of the system is invariant under time reversal, one should concentrate on events in which the PTRIV contribution is small, so that the genuine TRIV can be extracted. Figs. 23 and 25 are 3-dimensional plots of $R_{em,1}^{PTRIV}$ and $R_{em,2}^{PTRIV}$ for the β^+ decay process of the A=8 system. $R_{em,1}^{PTRIV}$ for the β^+ decay process, which contains the first forbidden contributions (namely contributions of order κ/M) obtained by Holstein [48], is of order $1 - 6 \times 10^{-4}$. $R_{em,2}^{PTRIV}$ for the β^+ decay process, which contains only second forbidden retardation terms (namely, terms of order κR with R some typical radius of a nucleus), is of order or less than 5×10^{-6} . Figs. 24 and 26 are 3-dimensional plots of $R_{em,1}^{PTRIV}$ and $R_{em,2}^{PTRIV}$ for the β^+ decay process of the A=8 system. $R_{em,1}^{PTRIV}$ for the β^- decay process is smaller than the corresponding one in the β^+ decay process due to the destructive interference of f_c and f_M . It is of order or less than 10^{-3} . $R_{em,2}^{PTRIV}$ for the β^- decay process depends only on the second

forbidden retardation terms. It is of order or less than 2×10^{-4} . From these graphs, it can be seen that the observables proportional to $\hat{\mathbf{k}}_\nu \hat{\mathbf{k}}_e \times \hat{\mathbf{k}}_\nu$ in both β^+ and β^- decay processes contain smaller leptonic Coulomb interaction induced PTRIV than other observables; thus it is advantageous to use these observables to study the question of genuine time reversal violation. The E_r dependence of $R_{em,i}^{PTRIV}$ (i=1,2) is determined by the energy dependence of $\eta_6, \eta_8, \dots, \eta_{10}$. As we can see, the PTRIV signal due to the e^\pm -nucleus Coulomb interaction reaches its maximum in the energy region $E_r \sim 3$ MeV, which is in contrast to the hadronic PTRIV. Therefore, although the e^\pm -nucleus Coulomb interaction is suppressed by a factor of $Z\alpha$, the magnitude of the PTRIV effects resulting from the hadronic strong interaction are smaller than those due to the e^\pm -nucleus Coulomb interaction in the energy region $E_r = 3$ MeV. Of course, the hadronic strong (and Coulomb) PTRIV can be bigger if one goes away from the $E_r = 3$ MeV region, in which different hadronic PTRIV amplitudes cancel each other.

6 Conclusions and Acknowledgments

In this paper, we have examined a time reversal test in the β -decays of the mass 8 nuclei. An R-matrix treatment was used, with parameters related to the α - α scattering phase shifts and allowed β -decay rates. The e^\pm -nucleus Coulomb final state interactions are included. False time-reversal violation (PTRIV) signals could arise due to these e^\pm -nucleus Coulomb as well as the strong interaction effects in the A=8 system. The strong interaction induced PTRIV is much smaller than the e^\pm -nucleus Coulomb interaction ones because the former is due to several second forbidden contributing terms that nearly cancel each other in the $E_r \approx 3$ –5 MeV region. In the energy region close to the lowest 2^+ state around 3 MeV these contributions can be further minimized because they have zeros. The e^\pm -nucleus final state interaction effects are $\lesssim 7 \times 10^{-4}$. These effects have not been seen so far.

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Appendix A: Hadronic Response Function in Terms of the Reduced Matrix Elements of Multipole Operators

Following Ref. [18],

$$R_{ss}(\sigma) = \sqrt{4\pi}\tilde{\sigma} \sum i^{J-J'} \left\{ \begin{matrix} J_f & J'_f & \sigma \\ J_i & J'_i & 0 \\ J & J' & \sigma \end{matrix} \right\} F_{ss}(JJ'; \sigma), \quad (\text{A.1})$$

$$R_{sv}^{(\pm)}(\sigma\rho) = \sqrt{4\pi}\tilde{\sigma} \sum i^{J-J'} \left\{ \begin{matrix} J_f & J'_f & \sigma \\ J_i & J'_i & 0 \\ J & J' & \sigma \end{matrix} \right\} F_{sv}^{(\pm)}(JJ'; \sigma\rho), \quad (\text{A.2})$$

$$R_{vv}(\sigma\rho\tau) = \sqrt{4\pi}\tilde{\sigma} \sum i^{J-J'} \left\{ \begin{matrix} J_f & J'_f & \sigma \\ J_i & J'_i & 0 \\ J & J' & \sigma \end{matrix} \right\} F_{vv}(JJ'; \sigma\rho\tau), \quad (\text{A.3})$$

where $\hat{x} = \sqrt{2x+1}$, in order to simplify the notation, a sum over indices not on the l.h.s. of the above equations is implied. It is found that

$$F_{ss}(JJ'; \sigma) = (-1)^J \frac{\langle \sigma || Y_{J'} || J \rangle}{\tilde{\sigma}} \mathcal{C}_J \mathcal{C}_{J'}^*, \quad (\text{A.4})$$

$$F_{sv}^{\pm}(JJ'; \sigma \rho) = f_{sv}(JJ'; \sigma \rho) \pm f_{vs}(JJ'; \sigma \rho), \quad (\text{A.5})$$

$$F_{vv}(JJ'; \sigma \rho \tau) = (-1)^{\rho} \tilde{J}^2 \tilde{J}'^2 \tilde{\tau} \sum_{l'} (-1)^{l+1} \langle \rho || Y_{l'} || l \rangle \left\{ \begin{matrix} 1 & l' & J' \\ 1 & l & J \\ \tau & \rho & \sigma \end{matrix} \right\} a_l^J a_{l'}^{J'*}, \quad (\text{A.6})$$

and

$$f_{sv}(JJ'; \sigma \rho) = (-1)^{\sigma} \tilde{J}'^2 \sum_l \langle \rho || Y_J || l \rangle \left\{ \begin{matrix} J & l & \rho \\ 1 & \sigma & J' \end{matrix} \right\} \mathcal{C}_J a_l^{J'*}, \quad (\text{A.7})$$

$$f_{vs}(JJ'; \sigma \rho) = (-1)^{J+1} \tilde{J}^2 \sum_l (-1)^l \langle \rho || Y_{J'} || l \rangle \left\{ \begin{matrix} J' & l & \rho \\ 1 & \sigma & J \end{matrix} \right\} a_l^J \mathcal{C}_{J'}^*, \quad (\text{A.8})$$

and

$$a_{J-1}^J = (-1)^J \frac{1}{2J+1} \left(\sqrt{J+1} \mathcal{E}_J + \sqrt{J} \mathcal{L}_J \right), \quad (\text{A.9})$$

$$a_J^J = (-1)^J \mathcal{M}_J, \quad (\text{A.10})$$

$$a_{J+1}^J = (-1)^J \frac{1}{2J+1} \left(\sqrt{J} \mathcal{E}_J - \sqrt{J+1} \mathcal{L}_J \right). \quad (\text{A.11})$$

The reduced matrix elements \mathcal{C}_J , \mathcal{L}_J , \mathcal{E}_J and \mathcal{M}_J of the hadronic charged weak currents can be decomposed into

$$\mathcal{C}_J = C_J + C_J^5 = \langle J_f || \hat{C}_J || J_i \rangle, \quad (\text{A.12})$$

$$\mathcal{L}_J = L_J + L_J^5 = \langle J_f || \hat{L}_J || J_i \rangle, \quad (\text{A.13})$$

$$\mathcal{E}_J = E_J + E_J^5 = \langle J_f || \hat{T}_J^{el} || J_i \rangle, \quad (\text{A.14})$$

$$\mathcal{M}_J = M_J + M_J^5 = \langle J_f || \hat{T}_J^{mag} || J_i \rangle, \quad (\text{A.15})$$

where superscript “5” indicates that the corresponding matrix element is originated from the axial vector current and the rest of the matrix elements are originated from the vector current.

Appendix B: Power Expansion of the Hadronic Response Functions for $A = 8$ System in Terms of Momentum Transfer

The relation between $R_1 \dots R_{12}$ and $R_{ss} \dots R_{vv}$ can be found. For T-even observables, only the final 2^+ state contributions are considered. While for T-odd observables, we consider contributions from the final 2^+ state, and from the interference between the final 2^+ state with the final 0^+ and 4^+ states. In the following discussion, we shall introduce set of response functions $\bar{R}_{ss}, \dots, \bar{R}_{vv}$ that are related to R_{ss}, \dots, R_{vv} through

$$\bar{R}_i = \frac{R_i}{|g_A|^2 |A_2|^2}, \quad (i = ss, \dots, vv) \quad (\text{B.1})$$

where A_2 is the Gamow–Teller matrix element defined in Table 3. With these set of response functions, the Cartesian ones can be found. The mass of the charged leptons m_e can be neglected; we do so in the following. Those correspond to $\sigma = 0$ and T-even observables are

$$\begin{aligned} R_1^{(0)} &= 5\bar{R}_{ss}(0) + 5\frac{\Delta}{\kappa}\bar{R}_{sv}^{(+)}(01) - 5\sqrt{3}\bar{R}_{vv}(000) \pm 5\sqrt{2}\frac{\epsilon_e - \epsilon_\nu}{\kappa}\bar{R}_{vv}(011) \\ &\quad + \frac{50}{3}\sqrt{\frac{2}{3}}\frac{\epsilon_e\epsilon_\nu}{\kappa^2}\bar{R}_{vv}(022), \end{aligned} \quad (\text{B.2})$$

$$\begin{aligned} R_2^{(0)} &= 5\bar{R}_{ss}(0) + 5\frac{\Delta}{\kappa}\bar{R}_{sv}^{(+)}(01) + \frac{5}{\sqrt{3}}\bar{R}_{vv}(000) \mp 5\sqrt{2}\frac{\epsilon_e - \epsilon_\nu}{\kappa}\bar{R}_{vv}(011) \\ &\quad + 10\sqrt{\frac{2}{3}}\frac{\epsilon_e^2 + \epsilon_\nu^2}{\kappa^2}\bar{R}_{vv}(022), \end{aligned} \quad (\text{B.3})$$

$$R_3^{(0)} = 5\sqrt{\frac{2}{3}}\frac{\epsilon_e\epsilon_\nu}{\kappa^2}\bar{R}_{vv}(022). \quad (\text{B.4})$$

Those correspond to $\sigma = 2$, T-even, which contain only the final 2^+ state contributions, are

$$R_1^{(2)} = -\frac{15}{\sqrt{7}}\frac{\epsilon_e^2}{\kappa^2}\bar{R}_{ss}(2) + \frac{15}{\sqrt{7}}\frac{\epsilon_e}{\kappa}\bar{R}_{sv}^{(+)}(21) \mp 5\sqrt{\frac{15}{7}}\frac{\epsilon_e\epsilon_\nu}{\kappa^2}\bar{R}_{sv}^{(-)}(22) \pm 15\sqrt{\frac{2}{7}}\frac{\epsilon_e}{\kappa}\bar{R}_{vv}(211)$$

$$+\frac{75}{2}\sqrt{\frac{6}{35}}\frac{\epsilon_e^2}{\kappa^2}\bar{R}_{vv}(220)+\frac{15}{7}\sqrt{30}\frac{\epsilon_e\epsilon_\nu}{\kappa^2}\bar{R}_{vv}(222), \quad (\text{B.5})$$

$$\begin{aligned} R_2^{(2)} = & -\frac{30}{\sqrt{7}}\frac{\epsilon_e\epsilon_\nu}{\kappa^2}\bar{R}_{ss}(2)+\frac{15}{\sqrt{7}}\frac{\Delta}{\kappa}\bar{R}_{sv}^{(+)}(21)\pm 5\sqrt{\frac{15}{7}}\frac{\epsilon_e^2-\epsilon_\nu^2}{\kappa^2}\bar{R}_{sv}^{(-)}(22) \\ & -10\sqrt{\frac{3}{7}}\bar{R}_{vv}(202)\mp 15\sqrt{\frac{2}{7}}\frac{\epsilon_e-\epsilon_\nu}{\kappa}\bar{R}_{vv}(211)+75\sqrt{\frac{6}{35}}\frac{\epsilon_e\epsilon_\nu}{\kappa^2}\bar{R}_{vv}(220) \\ & +\frac{5}{7}\sqrt{30}\frac{\epsilon_e^2+\epsilon_\nu^2}{\kappa^2}\bar{R}_{vv}(222), \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} R_3^{(2)} = & -\frac{15}{\sqrt{7}}\frac{\epsilon_\nu^2}{\kappa^2}\bar{R}_{ss}(2)+\frac{15}{\sqrt{7}}\frac{\epsilon_\nu}{\kappa}\bar{R}_{sv}^{(+)}(21)\pm 5\sqrt{\frac{15}{7}}\frac{\epsilon_e\epsilon_\nu}{\kappa^2}\bar{R}_{sv}^{(-)}(22) \\ & \mp 15\sqrt{\frac{2}{7}}\frac{\epsilon_\nu}{\kappa}\bar{R}_{vv}(211)+\frac{75}{2}\sqrt{\frac{6}{35}}\frac{\epsilon_\nu^2}{\kappa^2}\bar{R}_{vv}(220)+\frac{15}{7}\sqrt{30}\frac{\epsilon_e\epsilon_\nu}{\kappa^2}\bar{R}_{vv}(222), \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} R_4^{(2)} = & -\frac{15}{\sqrt{7}}\frac{\epsilon_e^2}{\kappa^2}\bar{R}_{ss}(2)\mp 5\sqrt{\frac{15}{7}}\frac{\epsilon_e^2}{\kappa^2}\bar{R}_{sv}^{(-)}(22)-\frac{25}{2}\sqrt{\frac{6}{35}}\frac{\epsilon_e^2}{\kappa^2}\bar{R}_{vv}(220) \\ & +\frac{5}{7}\sqrt{30}\frac{\epsilon_e^2}{\kappa^2}\bar{R}_{vv}(222), \end{aligned} \quad (\text{B.8})$$

$$R_5^{(2)} = -\frac{30}{\sqrt{7}}\frac{\epsilon_e\epsilon_\nu}{\kappa^2}\bar{R}_{ss}(2)-25\sqrt{\frac{6}{35}}\frac{\epsilon_e\epsilon_\nu}{\kappa^2}\bar{R}_{vv}(220)-\frac{10}{7}\sqrt{30}\frac{\epsilon_e\epsilon_\nu}{\kappa^2}\bar{R}_{vv}(222), \quad (\text{B.9})$$

$$\begin{aligned} R_6^{(2)} = & -\frac{15}{\sqrt{7}}\frac{\epsilon_\nu^2}{\kappa^2}\bar{R}_{ss}(2)\pm 5\sqrt{\frac{15}{7}}\frac{\epsilon_\nu^2}{\kappa^2}\bar{R}_{sv}^{(-)}(22)-\frac{25}{2}\sqrt{\frac{6}{35}}\frac{\epsilon_\nu^2}{\kappa^2}\bar{R}_{vv}(220) \\ & -\frac{5}{7}\sqrt{30}\frac{\epsilon_\nu^2}{\kappa^2}\bar{R}_{vv}(222). \end{aligned} \quad (\text{B.10})$$

Those correspond to $\sigma = 2$, T-odd, which contain only the final 2^+ state contributions are

$$\begin{aligned} R_7^{(2)} = & 15\sqrt{\frac{5}{21}}\frac{\epsilon_e(\epsilon_e-\epsilon_\nu)}{\kappa^2}Im\bar{R}_{sv}^{(+)}(22)\pm\frac{15}{\sqrt{7}}\frac{\epsilon_e}{\kappa}Im\bar{R}_{sv}^{(-)}(21) \\ & \mp 5\sqrt{\frac{30}{7}}\frac{\epsilon_e(\epsilon_e-\epsilon_\nu)}{\kappa^2}Im\bar{R}_{vv}(221)-5\sqrt{\frac{6}{7}}\frac{\epsilon_e}{\kappa}Im\bar{R}_{vv}(212), \end{aligned} \quad (\text{B.11})$$

$$\begin{aligned} R_8^{(2)} = & 15\sqrt{\frac{5}{21}}\frac{\epsilon_\nu(\epsilon_e-\epsilon_\nu)}{\kappa^2}Im\bar{R}_{sv}^{(+)}(22)\pm\frac{15}{\sqrt{7}}\frac{\epsilon_\nu}{\kappa}Im\bar{R}_{sv}^{(-)}(21) \\ & \mp 5\sqrt{\frac{30}{7}}\frac{\epsilon_\nu(\epsilon_e-\epsilon_\nu)}{\kappa^2}Im\bar{R}_{vv}(221)+5\sqrt{\frac{6}{7}}\frac{\epsilon_\nu}{\kappa}Im\bar{R}_{vv}(212). \end{aligned} \quad (\text{B.12})$$

Those correspond to $\sigma = 2$, T-odd, which originate from the interference contributions between final 2^+ state and final 0^+ state, are

$$\begin{aligned}
R_9^{(2)} = & \sin(\delta_2 - \delta_0) \left[15\sqrt{\frac{2}{3}} \frac{\epsilon_e(\epsilon_e - \epsilon_\nu)}{\kappa^2} \text{Re} \left(\bar{R}_{sv}^{(+)}(22) \right)_{20} \pm 15\sqrt{\frac{2}{5}} \frac{\epsilon_e}{\kappa} \text{Re} \left(\bar{R}_{sv}^{(-)}(21) \right)_{20} \right. \\
& \mp 10\sqrt{3} \frac{\epsilon_e(\epsilon_e - \epsilon_\nu)}{\kappa^2} \text{Re} \left(\bar{R}_{vv}(221) \right)_{20} - 10\sqrt{\frac{3}{5}} \frac{\epsilon_e}{\kappa} \text{Re} \left(\bar{R}_{vv}(212) \right)_{20} \left. \right] \\
& - \cos(\delta_2 - \delta_0) \left[15\sqrt{\frac{2}{3}} \frac{\epsilon_e(\epsilon_e - \epsilon_\nu)}{\kappa^2} \text{Im} \left(\bar{R}_{sv}^{(+)}(22) \right)_{20} \pm 15\sqrt{\frac{2}{5}} \frac{\epsilon_e}{\kappa} \text{Im} \left(\bar{R}_{sv}^{(-)}(21) \right)_{20} \right. \\
& \mp 10\sqrt{3} \frac{\epsilon_e(\epsilon_e - \epsilon_\nu)}{\kappa^2} \text{Im} \left(\bar{R}_{vv}(221) \right)_{20} - 10\sqrt{\frac{3}{5}} \frac{\epsilon_e}{\kappa} \text{Im} \left(\bar{R}_{vv}(212) \right)_{20} \left. \right], \quad (\text{B.13})
\end{aligned}$$

$$\begin{aligned}
R_{10}^{(2)} = & \sin(\delta_2 - \delta_0) \left[15\sqrt{\frac{2}{3}} \frac{\epsilon_\nu(\epsilon_e - \epsilon_\nu)}{\kappa^2} \text{Re} \left(\bar{R}_{sv}^{(+)}(22) \right)_{20} \pm 15\sqrt{\frac{2}{5}} \frac{\epsilon_\nu}{\kappa} \text{Re} \left(\bar{R}_{sv}^{(-)}(21) \right)_{20} \right. \\
& \mp 10\sqrt{3} \frac{\epsilon_\nu(\epsilon_e - \epsilon_\nu)}{\kappa^2} \text{Re} \left(\bar{R}_{vv}(221) \right)_{20} + 10\sqrt{\frac{3}{5}} \frac{\epsilon_\nu}{\kappa} \text{Re} \left(\bar{R}_{vv}(212) \right)_{20} \left. \right] \\
& - \cos(\delta_2 - \delta_0) \left[15\sqrt{\frac{2}{3}} \frac{\epsilon_\nu(\epsilon_e - \epsilon_\nu)}{\kappa^2} \text{Im} \left(\bar{R}_{sv}^{(+)}(22) \right)_{20} \pm 15\sqrt{\frac{2}{5}} \frac{\epsilon_\nu}{\kappa} \text{Im} \left(\bar{R}_{sv}^{(-)}(21) \right)_{20} \right. \\
& \mp 10\sqrt{3} \frac{\epsilon_\nu(\epsilon_e - \epsilon_\nu)}{\kappa^2} \text{Im} \left(\bar{R}_{vv}(221) \right)_{20} + 10\sqrt{\frac{3}{5}} \frac{\epsilon_\nu}{\kappa} \text{Im} \left(\bar{R}_{vv}(212) \right)_{20} \left. \right]. \quad (\text{B.14})
\end{aligned}$$

Those correspond to $\sigma = 2$, T-odd, which originate from the interference contributions between final 2^+ state and final 4^+ state, are

$$\begin{aligned}
R_{11}^{(2)} = & \sin(\delta_2 - \delta_4) \left[30\sqrt{\frac{3}{7}} \frac{\epsilon_e(\epsilon_e - \epsilon_\nu)}{\kappa^2} \text{Re} \left(\bar{R}_{sv}^{(+)}(22) \right)_{24} \pm 18\sqrt{\frac{5}{7}} \frac{\epsilon_e}{\kappa} \text{Re} \left(\bar{R}_{sv}^{(-)}(21) \right)_{24} \right. \\
& \mp 30\sqrt{\frac{6}{7}} \frac{\epsilon_e(\epsilon_e - \epsilon_\nu)}{\kappa^2} \text{Re} \left(\bar{R}_{vv}(221) \right)_{24} - 6\sqrt{\frac{30}{7}} \frac{\epsilon_e}{\kappa} \text{Re} \left(\bar{R}_{vv}(212) \right)_{24} \left. \right] \\
& - \cos(\delta_2 - \delta_4) \left[30\sqrt{\frac{3}{7}} \frac{\epsilon_e(\epsilon_e - \epsilon_\nu)}{\kappa^2} \text{Im} \left(\bar{R}_{sv}^{(+)}(22) \right)_{24} \pm 18\sqrt{\frac{5}{7}} \frac{\epsilon_e}{\kappa} \text{Im} \left(\bar{R}_{sv}^{(-)}(21) \right)_{24} \right.
\end{aligned}$$

$$\mp 30\sqrt{\frac{6}{7}}\frac{\epsilon_e(\epsilon_e - \epsilon_\nu)}{\kappa^2}Im\left(\bar{R}_{vv}(221)\right)_{24} - 6\sqrt{\frac{30}{7}}\frac{\epsilon_e}{\kappa}Im\left(\bar{R}_{vv}(212)\right)_{24}\Bigg], \quad (\text{B.15})$$

$$\begin{aligned} R_{12}^{(2)} = & \sin(\delta_2 - \delta_4) \left[30\sqrt{\frac{3}{7}}\frac{\epsilon_\nu(\epsilon_e - \epsilon_\nu)}{\kappa^2}Re\left(\bar{R}_{sv}^{(+)}(22)\right)_{24} \pm 18\sqrt{\frac{5}{7}}\frac{\epsilon_\nu}{\kappa}Re\left(\bar{R}_{sv}^{(-)}(21)\right)_{24} \right. \\ & \left. \mp 30\sqrt{\frac{6}{7}}\frac{\epsilon_\nu(\epsilon_e - \epsilon_\nu)}{\kappa^2}Re\left(\bar{R}_{vv}(221)\right)_{24} + 6\sqrt{\frac{30}{7}}\frac{\epsilon_\nu}{\kappa}Re\left(\bar{R}_{vv}(212)\right)_{24} \right] \\ & - \cos(\delta_2 - \delta_4) \left[30\sqrt{\frac{3}{7}}\frac{\epsilon_\nu(\epsilon_e - \epsilon_\nu)}{\kappa^2}Im\left(\bar{R}_{sv}^{(+)}(22)\right)_{24} \pm 18\sqrt{\frac{5}{7}}\frac{\epsilon_\nu}{\kappa}Im\left(\bar{R}_{sv}^{(-)}(21)\right)_{24} \right. \\ & \left. \mp 30\sqrt{\frac{6}{7}}\frac{\epsilon_\nu(\epsilon_e - \epsilon_\nu)}{\kappa^2}Im\left(\bar{R}_{vv}(221)\right)_{24} + 6\sqrt{\frac{30}{7}}\frac{\epsilon_\nu}{\kappa}Im\left(\bar{R}_{vv}(212)\right)_{24} \right], \quad (\text{B.16}) \end{aligned}$$

The subindices “20” and “24” in Eqs. B.13–B.16 denote they are the interference terms between the 2^+ state and 0^+ or 4^+ respectively. Response function R_{ss}, \dots, R_{vv} are related to the reduced matrix elements of the multipole operators of the hadronic charged weak currents; they are given in Appendix A. Using the definition for the static multipoles given in Tables 2 and 3, the response functions are ready to be expressed in terms of f_1, f_M, f_T, f_c^5 and η_k ($k = 1, \dots, 10$). (See Eqs. 2.41–2.45.) The resulting T-even response functions have the following form

$$\begin{aligned} R_1^{(0)} = & 1 + \frac{\Delta}{3M}Re\left(f_c^5 \mp 2f_M \mp 2f_T - \sqrt{\frac{3}{2}}\frac{\Delta}{M}\eta_7\right) \\ & + \frac{4\epsilon_e}{3M}Re\left(\pm f_M + \frac{5}{6}\sqrt{\frac{2}{3}}\frac{\Delta}{M}\eta_7 + \frac{\sqrt{2}}{9}\frac{\Delta}{M}\eta_8\right) \\ & - \frac{2\epsilon_e^2}{9M^2}Re\left(5\sqrt{\frac{2}{3}}\eta_7 + \frac{2\sqrt{2}}{3}\eta_8\right), \quad (\text{B.17}) \\ R_2^{(0)} = & -\frac{1}{3} + \frac{\Delta}{3M}Re\left(f_c^5 \pm 2f_M \pm 2f_T + \frac{1}{3}\sqrt{\frac{3}{2}}\frac{\Delta}{M}\eta_7 + \frac{2\sqrt{2}}{9}\frac{\Delta}{M}\eta_8\right) \\ & + \frac{4\epsilon_e}{3M}Re\left(\mp f_M - \sqrt{\frac{2}{3}}\frac{\Delta}{M}\eta_7 - \frac{2\sqrt{2}}{15}\frac{\Delta}{M}\eta_8\right) \end{aligned}$$

$$+\frac{4\epsilon_e^2}{9M^2}Re\left(\sqrt{6}\eta_7+\frac{2\sqrt{2}}{5}\eta_8\right), \quad (\text{B.18})$$

$$R_3^{(0)} = \frac{\Delta\epsilon_e}{9M^2}Re\left(\sqrt{6}\eta_7+\frac{2\sqrt{2}}{5}\eta_8\right)-\frac{\epsilon_e^2}{9M^2}Re\left(\sqrt{6}\eta_7+\frac{2\sqrt{2}}{5}\eta_8\right), \quad (\text{B.19})$$

$$R_1^{(2)} = \frac{\epsilon_e}{2M}Re\left(-f_c^5 \pm f_M \mp f_T \mp \frac{1}{3}\sqrt{\frac{3}{35}}\frac{\Delta}{M}f_1\eta_6 - \frac{\sqrt{2}}{5}\frac{\Delta}{M}\eta_8 + \frac{1}{5}\sqrt{\frac{10}{7}}\frac{\Delta}{M}\eta_9\right. \\ \left. + \frac{8}{35}\frac{1}{\sqrt{7}}\frac{\Delta}{M}\eta_{10}\right) + \frac{\epsilon_e^2}{M^2}Re\left(\mp\frac{1}{7}\sqrt{\frac{7}{15}}f_1\eta_6 - \frac{1}{5}\sqrt{\frac{10}{7}}\eta_9 + \frac{2}{7\sqrt{7}}\eta_{10}\right), \quad (\text{B.20})$$

$$R_2^{(2)} = -1 + \frac{\Delta}{2M}Re\left(-f_c^5 \pm f_M \pm f_T \mp \frac{1}{3}\sqrt{\frac{3}{35}}\frac{\Delta}{M}f_1\eta_6 + \frac{2\Delta}{3M}\eta_7 - \frac{\sqrt{2}}{15}\frac{\Delta}{M}\eta_8\right. \\ \left. + \frac{1}{15}\sqrt{\frac{10}{7}}\frac{\Delta}{M}\eta_9 + \frac{8}{105\sqrt{7}}\frac{\Delta}{M}\eta_{10}\right) + \frac{\epsilon_e}{M}Re\left(\mp f_M \pm \frac{1}{3}\sqrt{\frac{3}{35}}\frac{\Delta}{M}f_1\eta_6\right. \\ \left. - \frac{2\Delta}{3M}\eta_7 - \frac{2\sqrt{2}}{15}\frac{\Delta}{M}\eta_8 - \frac{4}{3}\sqrt{\frac{2}{35}}\frac{\Delta}{M}\eta_9 + \frac{76}{105\sqrt{7}}\frac{\Delta}{M}\eta_{10}\right) \\ \left. + \frac{\epsilon_e^2}{M^2}Re\left(\frac{2}{3}\eta_7 + \frac{2\sqrt{2}}{15}\eta_8 + \frac{4}{3}\sqrt{\frac{2}{35}}\eta_9 - \frac{76}{105\sqrt{7}}\eta_{10}\right), \quad (\text{B.21})$$

$$R_3^{(2)} = \frac{\Delta}{2M}\left(-f_c^5 \mp f_M \mp f_T \pm \sqrt{\frac{3}{35}}\frac{\Delta}{M}f_1\eta_6 - \frac{\sqrt{2}}{5}\frac{\Delta}{M}\eta_8 - \sqrt{\frac{2}{35}}\frac{\Delta}{M}\eta_9 + \frac{4}{5\sqrt{7}}\frac{\Delta}{M}\eta_{10}\right) \\ + \frac{\epsilon_e}{2M}Re\left(f_c^5 \pm f_M \pm f_T \mp \sqrt{\frac{5}{21}}\frac{\Delta}{M}f_1\eta_6 + \frac{\sqrt{2}}{5}\frac{\Delta}{M}\eta_8 + 3\sqrt{\frac{2}{35}}\frac{\Delta}{M}\eta_9\right. \\ \left. - \frac{48}{35\sqrt{7}}\frac{\Delta}{M}\eta_{10}\right) + \frac{(\Delta - \epsilon_e)^2}{M^2}Re\left(\pm\frac{1}{7}\sqrt{\frac{7}{15}}f_1\eta_6 - \sqrt{\frac{2}{35}}\eta_9 + \frac{2}{7\sqrt{7}}\eta_{10}\right), \quad (\text{B.22})$$

$$R_4^{(2)} = \frac{\epsilon_e^2}{M^2}Re\left(\pm\frac{1}{7}\sqrt{\frac{7}{15}}f_1\eta_6 + \frac{1}{15}\sqrt{\frac{10}{7}}\eta_9 - \frac{2}{21\sqrt{7}}\eta_{10}\right), \quad (\text{B.23})$$

$$R_5^{(2)} = \frac{\epsilon_e}{M}Re\left(\frac{2}{3}\frac{\Delta}{M}\eta_7 + \frac{2\sqrt{2}}{15}\frac{\Delta}{M}\eta_8 - \frac{12}{35\sqrt{7}}\frac{\Delta}{M}\eta_{10}\right)$$

$$+\frac{\epsilon_e^2}{M^2}Re\left(-\frac{2}{3}\eta_7-\frac{2\sqrt{2}}{15}\eta_8+\frac{12}{35\sqrt{7}}\eta_{10}\right), \quad (\text{B.24})$$

$$\begin{aligned} R_6^{(2)} &= \frac{\Delta^2}{M^2}Re\left(\mp\frac{1}{7}\sqrt{\frac{7}{15}}f_1\eta_6-\frac{\sqrt{2}}{15}\eta_8+\frac{6}{35\sqrt{7}}\eta_{10}\right) \\ &+ \frac{\Delta\epsilon_e}{3M^2}Re\left(\pm\frac{2}{7}\sqrt{\frac{7}{15}}f_1\eta_6-\frac{2\sqrt{2}}{5}\eta_8+\frac{12}{35\sqrt{7}}\eta_{10}\right) \\ &+ \frac{\epsilon_e^2}{M^2}Re\left(\mp\frac{1}{7}\sqrt{\frac{7}{15}}f_1\eta_6+\frac{\sqrt{2}}{15}\eta_8-\frac{6}{35\sqrt{7}}\eta_{10}\right). \end{aligned} \quad (\text{B.25})$$

Before ending this appendix, some more details are worth mentioning. The general form of response functions Eqs. B.2–B.10 depend on $\mathbf{k}_e \cdot \mathbf{k}_\nu$ though their dependence on κ . When the power expansion in terms of κ to order $O(\kappa^2)$ is performed, these dependence are extracted to make the response function defined in Eqs. B.17–B.25 independent of $\mathbf{k}_e \cdot \mathbf{k}_\nu$. Therefore $R_i^{(0)}$ ($i=1,2,3$) and $R_i^{(2)}$ ($i=1,\dots,6$) defined in Eqs. B.2–B.10 are slightly different from those defined in Eqs. B.17–B.25, which are independent of $\mathbf{k}_e \cdot \mathbf{k}_\nu$. However, the differences are small and only depend on η_7 .

Appendix C: Non-relativistic Reduction of the One Nucleon Matrix Elements of the Hadronic Weak Current Operators

$$\rho^{(\pm)}(\mathbf{x}) = F_1^V \sum_{i=1}^A \tau^{(\pm)}(i) \delta(\mathbf{x} - \mathbf{r}_i), \quad (\text{C.1})$$

$$\begin{aligned} \rho^{(\pm)5}(\mathbf{x}) &= \mp \frac{ig_T}{2M} \sum_{i=1}^A \tau^{(\pm)}(i) \sigma(i) \cdot \nabla \delta(\mathbf{x} - \mathbf{r}_i) \\ &+ g_A \sum_{i=1}^A \tau^{(\pm)}(i) \sigma(i) \cdot \left[\frac{\mathbf{p}_i}{2M} \delta(\mathbf{x} - \mathbf{r}_i) + \delta(\mathbf{x} - \mathbf{r}_i) \frac{\mathbf{p}_i}{2M} \right], \end{aligned} \quad (\text{C.2})$$

$$J^{(\pm)}(\mathbf{x}) = F_1^V \sum_{i=1}^A \tau^{(\pm)}(i) \left[\frac{\mathbf{p}_i}{2M} \delta(\mathbf{x} - \mathbf{r}_i) + \delta(\mathbf{x} - \mathbf{r}_i) \frac{\mathbf{p}_i}{2M} \right]$$

$$+\frac{G_M^V}{2M}\sum_{i=1}^A\tau^{(\pm)}(i)\nabla\times\sigma(i)\delta(\mathbf{x}-\mathbf{r}_i), \quad (\text{C.3})$$

$$A^{(\pm)}(\mathbf{x}) = \left(g_A \pm \frac{\Delta}{2M}g_T\right)\sum_{i=1}^A\tau^{(\pm)}(i)\sigma(i)\delta(\mathbf{x}-\mathbf{r}_i), \quad (\text{C.4})$$

where $\Delta = E_i - E_f$ is the difference of the initial and the final energies of the hadronic system, $G_M^V = F_M^V + 1 = 4.71$, terms that depend on g_P are not included since their contribution to the differential decay rate is extremely small when the charged lepton polarizations are not detected [20].

Appendix D: R-matrix theory for the β -decay processes of the $A = 8$ system

We only present an outline of the theory here; for more details we refer the reader to Refs. [19,26].

To the first order in the weak interaction Hamiltonian H_w and all orders in the strong and electromagnetic interaction Hamiltonian H_s and H_{em} , the S -matrix for the ${}^8\text{B}$ or ${}^8\text{Li} \rightarrow 2\alpha$ transition can be written as

$$S_{fi} = -i \int_{-\infty}^{\infty} dt \langle 2\alpha; e\nu | U(\infty, t) H_w U(t, -\infty) | \phi_i \rangle, \quad (\text{D.1})$$

where $U(t_1, t_2)$ is the full propagator of the system under strong and electromagnetic interactions and $|\phi_i\rangle$ is either ${}^8\text{B}$ or ${}^8\text{Li}$. With a proper phase convention for $|\phi_i\rangle$, which is regarded as an eigenstate of $H = T + H_s + H_{em}$ (T is the kinetic energy operator),

$$S_{fi} = -i \int_{-\infty}^{\infty} dt e^{-iE_i t} \langle 2\alpha; e\nu | U(\infty, t) H_w | \phi_i \rangle. \quad (\text{D.2})$$

A complete set of states can be inserted between the propagator and the weak interaction Hamiltonian H_w ,

$$S_{fi} = -i \int_{-\infty}^{\infty} dt e^{-iE_i t} \sum_I \langle 2\alpha; e\nu | U(\infty, t) | I \rangle \langle I | H_w | \phi_i \rangle, \quad (\text{D.3})$$

which is illustrated in Fig. 1. Since the matrix elements of the weak interaction Hamiltonian H_w are large only when the hadronic part of the state

$|I\rangle$ is comparable in spatial extent to that of $|\phi_i\rangle$ ^{D.1}, the summation in Eq. D.3 can be saturated to a satisfactory precision by a subset of states with comparable spatial extensions to $|\phi_i\rangle$. These sets of states can be generated by choosing eigenstates of a Hamiltonian that confine them in a region with a size that is comparable to $|\phi_i\rangle$. It is natural to choose the direct product of shell model states $|n\rangle$ and the leptonic states $|e\nu\rangle$. The shell model Hamiltonian is so constructed that the energies of its eigenstates are approximately located at the resonant peaks^{D.2} of the ${}^8\text{Be}^*$ system. At low energies, only a very few such shell model states are important, since the higher excitations of the shell model Hamiltonian have energies well separated from the ground state.

The strong and electromagnetic interaction Hamiltonian can be written as

$$H_s + H_{em} = H_{shell} + \Delta V, \quad (\text{D.4})$$

where H_{shell} is the shell model Hamiltonian and ΔV is the residual interaction that is responsible for the decay of the shell model ${}^8\text{Be}^*$ states. The decay of the localized state $|n\rangle$, of interest in this paper, to an asymptotic two α state can be parameterized in terms of the R -matrix [26] as

$$\begin{aligned} R_{2\alpha,n}(t) &= \langle 2\alpha; e\nu | U(\infty, t) | n \rangle, \\ &= \int_C \frac{dE}{2\pi i} e^{iEt} \frac{1}{E - E_{2\alpha}} \Gamma_{2\alpha,n}(E) \frac{1}{E - E_n - R_n(E)}, \end{aligned} \quad (\text{D.5})$$

where the path of the integration in the complex E plane is given in Fig. 4. The energy dependent function $\Gamma_{2\alpha,n}(E)$ is the vertex function for the decay of the shell model states $|n\rangle$ into the two α particle states and the energy dependent function $R_n(E)$ is the self energy of the state $|n\rangle$. They are all

^{D.1}It is possible that some states in the continuum also have large matrix elements. In case of the $A=8$ system, it does not turn out to be the case. It will be demonstrated in the following that the direct transition from ${}^8\text{B}$ or ${}^8\text{Li}$ to two α particle scattering states characterized by a parameter C is small.

^{D.2}i.e. the energies at which the phase shifts of the α - α scattering S -matrix pass through $(n + \frac{1}{2})\pi$ from below, with $n = 0, \pm 1, \dots$

related to ΔV in the following way

$$R_n(E) = \langle n | \Delta V F | n \rangle, \quad (\text{D.6})$$

$$\Gamma_{2\alpha,n}(E) = \langle 2\alpha | \Delta V F | n \rangle, \quad (\text{D.7})$$

where the state $F | n \rangle = | \psi_n \rangle$ satisfies a Lippmann–Schwinger type of equation

$$| \psi_n^\pm \rangle = | n \rangle + \frac{1}{E - H_{shell} \pm i\epsilon} \Delta V | \psi_n^\pm \rangle, \quad (\text{D.8})$$

with $| n \rangle$ an eigenstate of H_{shell} .

Figure Captions

Figure 1: Diagrammatic representation of the β -decay in A=8 system.

Figure 2: ${}^8\text{Be}$ energy levels calculated in the shell model compared with the experimental ones. The shell model energy level at 13.37 MeV should be identified with the experimental one at 16.92 MeV. It is explained in the text. The shell model energy level at 16.19 MeV should be identified with the experimental one at 16.63 MeV.

Figure 3: Experimental resonant levels in the ${}^8\text{Be}^*$ system with $J^P = 0^+$, 2^+ and 4^+ . The first column is the 0^+ series, the second column is the 2^+ series and the third column is the 4^+ series.

Figure 4: Result of the least square fit of the R-matrix parameters to the experimental phase shifts of the $J^P = 0^+$ state.

Figure 5: Result of the least square fit of the R-matrix parameters to the experimental phase shifts of the $J^P = 2^+$ state.

Figure 6: Result of the least square fit of the R-matrix parameters to the experimental phase shifts of the $J^P = 4^+$ state.

Figure 7: R-matrix calculation of the α particle spectrum in the β -decay process ${}^8\text{B} \rightarrow \alpha + \alpha + e^+ + \nu_e$. Solid curve represents the full calculation and dashed curve represents the calculation without the final potential scattering between the final two α particles. Open circles represent the experimental measured values. The contributions of the 3 MeV state and the 16 MeV doublet states to the total counts are also indicated.

Figure 8: R-matrix calculation of the α particle spectrum in the β -decay

process ${}^8\text{Li} \rightarrow \alpha + \alpha + e^- + \bar{\nu}_e$. Others are the same as Fig. 8.

Figure 9: R-matrix calculation of the α particle spectrum in the β -decay process ${}^8\text{B} \rightarrow \alpha + \alpha + e^+ + \nu_e$. A linear scale is used. Open circles represent the experimental measured values.

Figure 10: R-matrix calculation of the α particle spectrum in the β -decay process ${}^8\text{Li} \rightarrow \alpha + \alpha + e^- + \bar{\nu}_e$. Others are the same as Fig. 10.

Figure 11: Energy dependence of $\eta_6(2^+, 0^+)$.

Figure 12: Energy dependence of $\eta_9(2^+, 0^+)$.

Figure 13: Energy dependence of $\eta_6(2^+, 4^+)$.

Figure 14: Energy dependence of $\eta_9(2^+, 4^+)$.

Figure 15: 3-dimensional plot of $R_1^{PTRIV}(E_r, \epsilon_e)$ for the β^+ decay of A=8 system. The vertical scale is 10^{-5} .

Figure 16: 3-dimensional plot of $R_2^{PTRIV}(E_r, \epsilon_e)$ for the β^+ decay of A=8 system. The vertical scale is 10^{-5} .

Figure 17: 3-dimensional plot of $R_1^{PTRIV}(E_r, \epsilon_e)$ for the β^- decay of A=8 system. The vertical scale is 10^{-5} .

Figure 18: 3-dimensional plot of $R_2^{PTRIV}(E_r, \epsilon_e)$ for the β^- decay of A=8 system. The vertical scale is 10^{-5} .

Figure 19: Averaged PTRV response function $\bar{R}_1^{PTRV}(E_r)$ as an function of E_r . Solid lines represent $\bar{R}_9^{(2)} + \bar{R}_{11}^{(2)}$. Dashed lines represent $\bar{R}_9^{(2)}$ and $\bar{R}_{11}^{(2)}$ for the β^- decay. Dash-dotted line represent $\bar{R}_9^{(2)}$ and $\bar{R}_{11}^{(2)}$ for the β^+ decay.

Figure 20: Averaged PTRV response function $\bar{R}_2^{PTRV}(E_r)$ as an function of E_r . Solid lines represent $\bar{R}_{10}^{(2)} + \bar{R}_{12}^{(2)}$. Dashed lines represent $\bar{R}_{10}^{(2)}$ and $\bar{R}_{12}^{(2)}$ for the β^- decay. Dash-dotted line represent $\bar{R}_{10}^{(2)}$ and $\bar{R}_{12}^{(2)}$ for the β^+ decay.

Figure 21: 3-dimensional plot of $R_{em,1}^{PTRV}(E_r, \epsilon_e)$ for the β^+ decay of A=8 system. The vertical scale is 10^{-4} .

Figure 22: 3-dimensional plot of $R_{em,1}^{PTRV}(E_r, \epsilon_e)$ for the β^- decay of A=8 system. The vertical scale is 10^{-4} .

Figure 23: 3-dimensional plot of $R_{em,2}^{PTRV}(E_r, \epsilon_e)$ for the β^+ decay of A=8 system. The vertical scale is 10^{-5} .

Figure 24: 3-dimensional plot of $R_{em,2}^{PTRV}(E_r, \epsilon_e)$ for the β^- decay of A=8 system. The vertical scale is 10^{-4} .

Figure 25: The path for the complex E integration. Solid dots represent poles of the integrand and the dashed line represents the cut of the integrand. (It is usually located on the real axis extending all the way to positive infinity. Two cuts along different directions are equivalent provided that there are no singularities in between them.)

Figure 26: Diagrammatic representation of the resonant level contribution to the α - α scattering amplitude.

Table 1: Kinematic functions considered. Here “T” and “P” denote the time reversal and parity reflection transformations.

$\sigma = 0$		
$\begin{smallmatrix} T \\ P \end{smallmatrix}$	Even	Odd
Even	$K_{ss}(0) K_{sv}^{(+)}(01) K_{vv}(000) K_{vv}(011)$ $K_{vv}(022)$	X
Odd	X	X
$\sigma = 2$		
$\begin{smallmatrix} T \\ P \end{smallmatrix}$	Even	Odd
Even	$K_{ss}(2) K_{sv}^{(+)}(21) K_{sv}^{(-)}(22) K_{vv}(211)$ $K_{vv}(202) K_{vv}(220) K_{vv}(222)$	X
Odd	X	$K_{sv}^{(+)}(22) K_{sv}^{(-)}(21) K_{vv}(212) K_{vv}(221)$

Table 2: Expansion of the reduced matrix elements of the non-relativistic hadronic one body weak current multipole operators in terms of power series of the momentum transfer $\kappa = |\mathbf{k}_e + \mathbf{k}_\nu|$. Here $\tilde{g}_A = g_A \pm \frac{\Delta}{2M} g_T$.

	Vector Current		Axial Vector Current
C_0	$F_1 \left(A_1 - \frac{1}{6} \frac{\kappa^2}{M^2} A_5 \right)$	C_1^5	$-i \frac{\kappa}{2M} \left(\frac{1}{\sqrt{3}} (g_A \mp g_T) A_2 + \frac{2}{3} g_A A_3 \right)$
L_0	$\frac{\Delta}{\kappa} F_1 \left(A_1 - \frac{1}{6} \frac{\kappa^2}{M^2} A_5 \right)$	L_1^5	$\frac{i}{\sqrt{3}} \tilde{g}_A \left(A_2 - \frac{\kappa^2}{6M^2} A_7 + \frac{\sqrt{2}\kappa^2}{15M^2} A_8 \right)$
M_1	$\frac{i\kappa}{2M} G_M \sqrt{\frac{2}{3}} A_2 - \frac{i\kappa}{3M} F_1 A_4$	E_1^5	$\frac{i\sqrt{2}}{\sqrt{3}} \tilde{g}_A \left(A_2 - \frac{\kappa^2}{6M^2} A_7 - \frac{\sqrt{2}\kappa^2}{30M^2} A_8 \right)$
C_2	$\frac{1}{15} \frac{\kappa^2}{M^2} F_1 A_6$	M_2^5	$\frac{1}{15} g_A \frac{\kappa^2}{M^2} A_9$
L_2	$\frac{1}{15} \frac{\kappa \Delta}{M^2} F_1 A_6$	E_3^5	$\frac{2i}{15\sqrt{7}} g_A \frac{\kappa^2}{M^2} A_{10}$
E_2	$\frac{1}{15} \sqrt{\frac{3}{2}} \frac{\kappa \Delta}{M^2} F_1 A_6$	L_3^5	$\frac{i}{15} \sqrt{\frac{3}{7}} g_A \frac{\kappa^2}{M^2} A_{10}$

Table 3: Definitions for the static multipole operators. Here $\langle J_f || \dots || J_i \rangle$ denotes the reduced matrix elements of the corresponding multipole operators. Vector σ represents the collection of three Pauli 2×2 matrices.

A_1	$\sum_{i=1}^A \langle J_f \tau^{(\pm)} Y_0 J_i \rangle$	A_6	$M^2 \sum_{i=1}^A \langle J_f \tau^{(\pm)} r^2 Y_2 J_i \rangle$
A_2	$\sum_{i=1}^A \langle J_f \tau^{(\pm)} \mathcal{Y}_{10} \cdot \sigma J_i \rangle$	A_7	$M^2 \sum_{i=1}^A \langle J_f \tau^{(\pm)} r^2 \mathcal{Y}_{10} \cdot \sigma J_i \rangle$
A_3	$\sum_{i=1}^A \langle J_f \tau^{(\pm)} r Y_1 \sigma \cdot \nabla J_i \rangle$	A_8	$M^2 \sum_{i=1}^A \langle J_f \tau^{(\pm)} r^2 \mathcal{Y}_{12} \cdot \sigma J_i \rangle$
A_4	$\sum_{i=1}^A \langle J_f \tau^{(\pm)} r \mathcal{Y}_{11} \cdot \nabla J_i \rangle$	A_9	$M^2 \sum_{i=1}^A \langle J_f \tau^{(\pm)} r^2 \mathcal{Y}_{22} \cdot \sigma J_i \rangle$
A_5	$M^2 \sum_{i=1}^A \langle J_f \tau^{(\pm)} r^2 Y_0 J_i \rangle$	A_{10}	$M^2 \sum_{i=1}^A \langle J_f \tau^{(\pm)} r^2 \mathcal{Y}_{32} \cdot \sigma J_i \rangle$

Table 4: R-matrix parameters for the $J^P = 0^+$ state obtained from a best fit to the experimental α - α scattering phase shifts.

E (MeV)	a MeV ⁻¹	b MeV ⁻²	c MeV ⁻¹	w_0 (MeV)	r_0 (fm)	z MeV ⁻²
0.0	0.324	-0.272	-6.622	0.011	4.39	1.34×10^{-4}

Table 5: R-matrix parameters for the $J^P = 2^+$ state obtained from a best fit to the experimental α - α scattering phase shifts. No energy dependence for the doublet states located near $E \sim 16$ MeV is considered, which is indicated by 0's in the table. The α - α phase shifts are very insensitive to the values of w_0 for the 16 MeV doublet states in the energy region considered, they will be determined in the α particle spectra fit.

E MeV	a MeV ⁻¹	b MeV ⁻²	c MeV ⁻¹	w_0 (MeV)	r_0 (fm)	z MeV ⁻²
3.36	0.145	-0.01704	0.00216	1.80		
16.63	0	0	0	?	2.45	5.2×10^{-4}
16.92	0	0	0	?		

Table 6: R-matrix parameters for the $J^P = 4^+$ state obtained from a best fit to the experimental α - α scattering phase shifts.

E MeV	a MeV^{-1}	b MeV^{-2}	c MeV^{-1}	w_0 (MeV)	r_0 (fm)	z MeV^{-2}
12.15	-0.400	-0.048	-0.126	3.54	3.65	0.0

Table 7: Values for various static single nucleon weak current form factors used.

g_A	F_1	G_M^V	g_T
-1.254	1.00	4.71	0.00

Table 8: Kinematic functions correspond to the contribution of the symmetric (with respect to $A^{\mu\nu}$ and $B^{\mu\nu}$) and dispersive parts of the final lepton–hadron Coulomb scattering T–matrix contribution.

$\sigma = 2$		
$\begin{smallmatrix} T \\ P \end{smallmatrix}$	Odd	Even
Odd	$K_{ss}(2) K_{sv}^{(+)}(21) K_{sv}^{(-)}(22) K_{vv}(211)$ $K_{vv}(202) K_{vv}(220) K_{vv}(222)$	X
Even	X	$K_{sv}^{(+)}(22) K_{sv}^{(-)}(21) K_{vv}(212) K_{vv}(221)$